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Modelling liquidity and the valuation of American options using the dual method

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Modelling Liquidity and the Valuation of American Options using the Dual Method

submitted by

Surbjeet Singh

for the degree of Doctor of Philosophy

of the

University of Bath

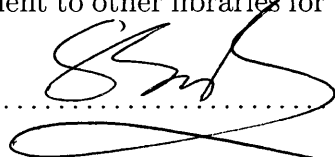
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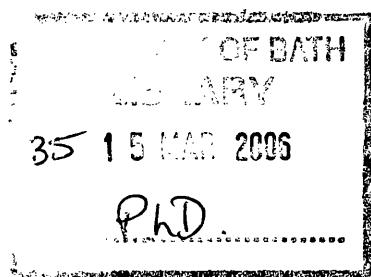
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Summary

The major part of this thesis is concerned with the problem of modelling liquidity. We introduce a discrete time model which takes into account the effect of volume of a risky asset traded on the price. We take the continuous time limit to arrive at modified wealth dynamics. We then consider one of the classic problems from the world of finance-the Merton problem of optimal investment and consumption with an infinite horizon under these new dynamics. We solve the problem numerically using a Markov chain approximation method. We also solve a simplified control problem in closed form which we verify numerically.

Another classic problem in the world of finance is option pricing. It is well known that in a complete market the payoff of an option can be replicated. This is not the case with an illiquid market. We study the problem from the point of view of utility indifference pricing. We provide numerical results and an expression for the asymptotic control.

Finally we consider the problem of Monte Carlo valuation of American options using the dual technique recently introduced in Rogers (2002) and Haugh and Kogan (2001). We look at the American put on a single asset and the American min-put on 2 assets. We refine the martingales used in Rogers (2002) to get better hedges.

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Chapter 1

Modelling Liquidity

1.1 Introduction

In the famous papers of Merton (1969), Black & Scholes (1973) and many others since then the assumption is made that stocks can be bought and sold in unlimited quantities at a price given by a geometric brownian motion. Of course, this is a simplification of reality but it gives useful results. We would like to introduce liquidity risk into the modelling framework, but firstly we need to decide what liquidity risk actually is. It is an ill-defined term and it is to do with the effect of trading “large quantities”. We can decompose it into two main features, price manipulation and transaction costs.

There is evidence that the actions of a large trader can influence the price of the underlying (see, for example, Jarrow (1992) and Hull (1997)). A large trader can try and “corner the market”. A way to do this in a commodity market is to take a huge long futures position and at the same time buy up the underlying commodity. As expiry approaches the investors who are short the futures contract may find that there is not enough supply to meet their demand and hence the price is pushed up. One such alleged case of this was the activities of the Hunt brothers in the silver market in 1979-1980. Their trading caused the price to increase from \$9 per ounce to \$50 per ounce.

Another way an agent can “corner the market”, this time involving shares, is to buy up a large supply and then lend some to investors who want to go short. When these shares are sold on the market the agent buys them up and then calls in the short shares. Since the agent has limited the supply of shares by buying

a large amount this pushes the price up. These types of manipulation involve taking huge positions in the underlying and documented examples of this type of activity have shown large price increases followed by a crash.

The second effect is best illustrated with reference to the market microstructure. Most modern markets are electronic order book driven markets (an order book is a list of unexecuted trades). For example, in London all the FTSE 100 shares and most of the FTSE 250 shares are traded on the Stock Exchange Electronic Trading Service (SETS). A trader either places a limit order or a market order. A limit order is an order to trade a specific quantity of shares at a specific price. If the limit order cannot be matched with existing orders in the book it is added to the book so in this way you are providing liquidity. Alternatively, a trader can place a market order which is an order to trade at the best available price. The buy/sell order is matched up with the lowest/highest price sell/buy order in the book. Of course, if the market order is large the trader will not receive the same price for each share of the order. The order may eat into higher or lower tiers of the limit order book. This is the second effect of (lack of) liquidity. The average share price paid/received is an increasing/decreasing function of the amount bought/sold.

1.2 Pure Feedback models

The first effect, where the price of the underlying can be manipulated, we call a feedback effect. Models of liquidity which only have this feature include Frey (1998), Frey (2000), Frey & Stremme (1997), Platen & Schweizer (1998), Schönbucher and Wilmott (2000) and Papanicolaou and Sircar (1998). Frey and Stremme (1997) starts off as a discrete time model and the limit is taken to arrive at a continuous time model.

The other models start off in continuous time. The general model framework is as follows. Instead of simply imposing a price process for the risky asset, an equilibrium argument is used to arrive at a price. There are many small traders and program (large) traders. The small traders are modelled by a representative reference trader and the program traders are modelled by a representative program trader, so we have just two traders in the model. The demand of the reference trader is $D(t, Y_t, x)$ where t is the current time, Y_t is the current value of

some state variable process and x is the stock price. The demand of the program trader is α . Normalizing the supply of the risky asset to 1 gives the equilibrium price, X_t , from the market clearing condition

$$D(t, Y_t, X_t) + \alpha = 1 \quad (1.2.1)$$

Frey and Stremme (1997) and Papanicolaou and Sircar (1998) consider the situation $\alpha = \rho \phi(t, X_t)$ for $\rho \geq 0$. They assume that for fixed t and Y_t there is a unique solution for X_t and for fixed t and X_t there is a unique solution for Y_t . It is assumed that Y_t follows a geometric Brownian motion i.e.

$$dY_t = Y_t(\eta dW_t + \mu dt) \quad (1.2.2)$$

where η and μ are constants.

Following Papanicolaou and Sircar (1998) it is quite straightforward to derive the dynamics of X_t . Writing the unique solution to (1.2.1) as $X_t = \psi(t, Y_t)$ and by applying Ito's formula we have

$$dX_t = Y_t \eta \psi_y(t, Y_t) dW_t + \left(\psi_t(t, Y_t) + Y_t \mu \psi_y(t, Y_t) + \frac{1}{2} Y_t^2 \eta^2 \psi_{yy}(t, Y_t) \right) dt \quad (1.2.3)$$

Writing $G(t, Y_t, X_t) = D(t, Y_t, X_t) + \rho \phi(t, X_t)$ we have

$$\begin{aligned} 0 &= dG \\ &= (G_x Y_t \eta \psi_y + G_y Y_t \eta) dW_t + \left(G_x (\psi_t + Y_t \mu \psi_y + \frac{1}{2} Y_t^2 \eta^2 \psi_{yy}) \right. \\ &\quad \left. + G_y Y_t \mu + G_t + \frac{1}{2} Y_t^2 \eta^2 (G_{yy} + \psi_y^2 G_{xx} + 2 \psi_y G_{xy}) \right) dt \end{aligned} \quad (1.2.4)$$

From which we get

$$dX_t = -Y_t \eta \frac{G_y}{G_x} dW_t - \left(Y_t \mu \frac{G_y}{G_x} + \frac{G_t}{G_x} + \frac{1}{2} Y_t^2 \eta^2 \left(\frac{G_{yy}}{G_x} + \frac{G_y^2 G_{xx}}{G_x^3} - 2 \frac{G_y G_{xy}}{G_x^2} \right) \right) dt \quad (1.2.5)$$

In Schönbucher and Wilmott (2000) the supply as well as the demand of the small traders is modelled. They define the excess demand $\mathcal{X}(t, W_t, X_t) =$

$\mathcal{D}(t, W_t, X_t) - \mathcal{S}(t, W_t, X_t)$ where $\mathcal{D}(t, W_t, X_t)$ is the aggregate demand for the risky asset of the small traders and $\mathcal{S}(t, W_t, X_t)$ is the aggregate supply of the risky asset from the small traders and W_t is a Brownian motion. They make the assumption that $\mathcal{X}_x(t, W_t, X_t) < 0$ which is reasonable from an economic point of view. As the price goes up people want to supply more and demand less. As the price goes down people demand more and supply less.

In Frey and Stremme (1997) and Papanicolaou and Sircar (1998) $D(t, Y_t, X_t) = \beta \frac{Y_t}{X_t}$ and $D(t, Y_t, X_t) = \beta \frac{Y_t^\gamma}{X_t}$ respectively where $\beta > 0$ so that $D_x < 0$. In Platen and Schweizer (1998) the market clearing condition is

$$U_t + \gamma \log\left(\frac{X_t}{X_0}\right) + \alpha(t, X_t) = k. \quad (1.2.6)$$

where U_t is an arithmetic Brownian motion with constant drift and volatility and k is a constant. This corresponds to a model using $D(t, Y_t, X_t) = \log(\beta Y_t X_t^\gamma)$. They use this demand function with $\gamma > 0$ so that demand increases with rising prices! In the model of Frey (2000) the dynamics of the risky asset is

$$dX_t = X_t(\sigma dW_t + \rho d\alpha(t, X_t)) \quad (1.2.7)$$

where $\rho > 0$. This corresponds to $D(t, Y_t, X_t) = \log(\beta Y_t X_t^\gamma)$ with $\gamma = -\frac{1}{\rho}$ i.e. it is the model of Platen and Schweizer (1998) with γ having the opposite sign.

The next logical step is to find option prices for these feedback models. However, there is an economic problem with trying to justify the existence of an option market if (1.2.1) is used as a model. This point is gone into detail in Schöbucher and Wilmott (2000). Suppose we have a single large trader then the market clearing condition (1.2.1) basically allows the large trader to manipulate the price X_t to any desired level by holding an appropriate amount of the risky asset. Imagine a large trader who sells a put or a call option. Just before expiry he could manipulate the share price so that the option is out of the money and back again just after. By doing this he avoids having to pay out. If the small traders are aware of the presence of the large trader they would refuse to buy an option from the large trader and the options market would collapse. In reality, of course, a large trader doesn't have complete power to manipulate the price and manipulation does incur some risk.

Despite this major failing of the model you can still calculate option prices from the point of view of the small trader. Assume the large trader follows a strategy of the form $\alpha(t, X_t) = \rho\phi(t, X_t)$. From (1.2.1) we arrive at the dynamics of X_t . The small traders then price an option in the usual way, e.g. see Schönbucher and Wilmott (2000). Suppose the portfolio of the small trader, π_t , is made up of $\Delta(t, X_t)$ of the risky asset and $c(t, X_t)$ of the riskless bond, B_t , and minus a single option worth $P(t, X_t)$, i.e.

$$\pi_t = \Delta(t, X_t)X_t + c(t, X_t)B_t - P(t, X_t) \quad (1.2.8)$$

The self financing condition implies

$$\begin{aligned} d\pi_t &= \Delta dX_t + r(\pi_t - \Delta(t, X_t)X_t + P(t, X_t))dt - dP_t \\ &= \Delta dX_t + r(\pi_t - \Delta(t, X_t)X_t + P(t, X_t))dt - (P_t dt + P_x dX_t + \frac{1}{2}P_{xx}dX_t \cdot dX_t) \end{aligned} \quad (1.2.9)$$

Choosing $\Delta(t, X_t) = P_x$ we can make the change in the portfolio riskless hence it must earn the riskfree rate i.e. $d\pi_t = r\pi_t dt$ so we have

$$r(\pi_t - P_x X_t + P(t, X_t))dt - (P_t dt + \frac{1}{2}P_{xx}dX_t \cdot dX_t) = r\pi_t dt \quad (1.2.10)$$

which gives using (1.2.5)

$$\frac{1}{2}\sigma^2(t, X_t)P_{xx}(t, X_t) + rX_t P_x(t, X_t) + P_t(t, X_t) = rP(t, X_t) \quad (1.2.11)$$

where $\sigma(t, X_t) = -Y_t \eta \frac{G_y}{G_x}$.

If the large trader wants to replicate the option payoff using a self financing strategy of the form $\alpha(t, X_t)$ you can do the calculation and it leads to a non-linear pde which is derived in Papanicolaou and Sircar (1998) and Schönbucher and Wilmott (2000). Frey (1998) derives a quasi-linear pde satisfied by the hedging strategy and proves existence and uniqueness.

The argument in the derivation of the non-linear pde is much like the standard

one but since $\Delta(t, X_t) = P_x$ using (1.2.1) we have the nonlinear pde.

$$\frac{1}{2}Y_t^2\eta^2\left(\frac{D_y}{D_x + P_{xx}}\right)^2P_{xx}(t, X_t) + rX_tP_x(t, X_t) + P_t(t, X_t) = rP(t, X_t) \quad (1.2.12)$$

1.3 Transaction Costs

The proportional transaction model of Magill and Constantinides (1976) is an attempt to model the bid-offer spread (the difference between the lowest sell price and the highest buy price in a limit order book) but their model does not take into account buying or selling such large quantities that you face more than one price.

Following Davis and Norman (1990) the model specification is as follows. The portfolio, π_t , has to be split into wealth in the risky asset and wealth in the cash bond since transfers between the two are no longer costless but subject to proportional transaction costs. Let π_t^0 be the wealth in the risky asset and π_t^1 the wealth in the cash bond. The wealth in the risky asset and cash respectively evolves as

$$\begin{aligned} d\pi_t^0 &= \pi_t^0(\sigma dW_t + \nu dt) + dL_t - dU_t \\ d\pi_t^1 &= r\pi_t^1 dt - (1 + \lambda)dL_t + (1 - \mu)dU_t \end{aligned} \quad (1.3.1)$$

where $\lambda, \nu, \mu > 0$.

L_t and U_t are cumulative purchases and sales of the risky asset on the interval $[0, t]$.

The transaction costs are captured by this formulation since an increase of dL_t of wealth in the risky asset is accompanied by a decrease in wealth in the cash bond of $(1 + \lambda)dL_t$ and a decrease in wealth of dU_t in the risky asset is accompanied by an increase in wealth in the cash bond of only $(1 - \mu)dU_t$. This captures the bid-offer spread effect but the transaction costs are proportional to the amount bought or sold so that you effectively have only one price for selling and one for buying. We would like to capture the effect of increasing prices the more you buy, and decreasing prices the more you sell.

Cetin, Jarrow and Protter (2004) attempt to model this aspect of liquidity. They postulate the existence of a supply curve. This is a function $S(t, x, \omega)$

which is the risky asset price per unit at time t for an order of size x for a given state $\omega \in \Omega$. $S(t, 0, \omega)$ is assumed to be a continuous semi-martingale and apart from some other technical assumptions the economic one is that $S(t, x, \omega)$ is non-decreasing in x . The notional wealth (suppressing the dependence on ω) is defined as $\pi_t = H_t S(t, 0) + C_t$ where H_t is the holding of the risky asset and C_t is the cash holding. As a result of their formulation the usual self financing condition is modified. Note that the notional wealth is not the real wealth in the sense that it cannot be turned into the equivalent amount of cash. It is just a book value.

Following Cetin et al. (2004) a heuristic derivation is as follows. Assuming H_t is a continuous semi-martingale

$$\begin{aligned}
dC_t &= -S(t+dt, dH_t)dH_t \\
&= -S(t, 0)dH_t - (S(t+dt, dH_t) - S(t+dt, 0))dH_t \\
&\quad - (S(t+dt, 0) - S(t, 0))dH_t \\
&= -S(t, 0)dH_t - (S(t+dt, dH_t) - S(t+dt, 0))dH_t - dS(t, 0).dH_t
\end{aligned} \tag{1.3.2}$$

Doing a Taylor expansion on the second term gives

$$dC_t = -S(t, 0)dH_t - S_x(t, 0)dH_t.dH_t - dS(t, 0).dH_t \tag{1.3.3}$$

Using

$$d(H_t S(t, 0)) = H_t dS(t, 0) + S(t, 0)dH_t + dS(t, 0).dH_t \tag{1.3.4}$$

$$d\pi_t = H_t dS(t, 0) - S_x(t, 0)dH_t.dH_t \tag{1.3.5}$$

If we allow H_t to have jumps (assume H_t is cadlag) then we have

$$d\pi_t = H_{t-} dS(t, 0) - \Delta H_t (S(t, \Delta H_t) - S(t, 0)) - S_x(t, 0)dH_t^c.dH_t^c \tag{1.3.6}$$

The first term is the usual one in the standard theory, the second is due to jumps in H_t and the third is due to the supply curve. The problem with this approach is that a continuous trading strategy of finite variation can avoid the liquidity costs since then $dH_t^c.dH_t^c$ is zero and ΔH_t is zero.

Longstaff (2001) is an interesting model which although doesn't model transaction costs explicitly, the trading strategies which are allowed are restricted. He assumes the trading strategy, H_t , is not only of finite variation but is differentiable with respect to time, and that derivative is bounded. This modelling idea is interpreting liquidity as a "thin" market so that there are limited number of shares to trade at any given time.

1.4 Mixed Models

Bank and Baum (2004) is a model which contains a feedback effect and incorporates a transaction cost element. They consider a family of continuous semimartingales, $P^\nu = (P_t^\nu)_{0 \leq t \leq T}$, indexed by a parameter ν where ν represents a constant stake of ν shares of the risky asset. If the investor has a time varying strategy the price evolution process is $P(t, \theta_t) = P_t^{\theta_t}$. Their framework includes the pure feedback models of Frey and Stremme (1997), Papanicolaou and Sircar (1998) and Schönbucher and Wilmott (2000). In deriving the wealth dynamics they assume that asset prices are affected by the large trader's order before the transaction occurs. This leads to transaction costs which depends on the quadratic variation of θ_t in a similar way to Cetin et al. (2004). This model therefore has the same drawback, namely, that transaction costs can be avoided by following a continuous trading strategy of finite variation.

Bakstein and Howison (2003) is a model in discrete time with observable parameters. It is based on modifying the standard Cox-Ross-Rubinstein (CRR) binomial tree. In the standard CRR binomial tree, If the share price at time t_i is S_{t_i} then at the next time step the share price can be $S_{t_{i+1}} = uS_{t_i}$ with probability q and $S_{t_{i+1}} = dS_{t_i}$ with probability $1 - q$. In the model of Bakstein and Howison (2003), if S_{t_i} is the share price at time t_i then the average share price for purchasing ΔH shares is given by $\tilde{S}_{t_i} = S_{t_i}(1 + \gamma \text{sign}(\Delta H))e^{\lambda \Delta H}$ where $\gamma, \lambda > 0$. γ captures the bid-offer spread and λ captures the idea that the average share price is an increasing function of the quantity bought. They also incorporate feedback using a parameter $0 \leq \alpha \leq 1$. The share price in the next time period is $uS_{t_i}^\alpha \tilde{S}_{t_i}^{1-\alpha}$ with probability q and $dS_{t_i}^\alpha \tilde{S}_{t_i}^{1-\alpha}$ with probability $1 - q$. The values of λ , γ and α are supposed to be found from studying the limit order book. In taking the continuous time limit of this formulation they set $\gamma = 0$ and assuming that

one starts from a hedged position and that the trading strategy is a function of the share price and time they derive a non-linear pde satisfied by an option price.

Our view is that we are not trying to model the extreme situation where a larger trader can buy up virtually the whole supply of the underlying. We are trying to model a share market where the large trader buys/sells amounts daily which are comparable to the size of the limit order book on a typical day, but the trade size is small compared to the total number of shares actually in supply. He therefore feels the transaction cost effect of liquidity but does not have much impact on the trades entering the limit order book after his transactions. Any impact is so small we ignore it and let the underlying share price follow an exogenous process. In the next section we introduce the discrete time model featuring a large trader. We then take the continuous time limit to derive new wealth evolution dynamics.

1.5 The Discrete Time Model

We consider a single risky asset whose notional price at time t is denoted by S_t . We divide time into equal time steps of size Δt . Let p_n denote $\log S_{n\Delta t}$, which we suppose evolves as a random walk,

$$p_n = p_{n-1} + \xi_n, \quad (1.5.1)$$

where the ξ_n are independent and identically distributed with mean and variance proportional to Δt .

In each period the supply of the small traders is $\Delta t f_s(\tilde{p}_n - p_n)$ where the supply function per unit time, f_s , is continuous and strictly increasing and \tilde{p}_n is the log price at which trades are executed. The demand of the small traders is $\Delta t f_d(\tilde{p}_n - p_n)$ where the demand function per unit time, f_d , is continuous and strictly decreasing. A hedger also comes to the market with the intention of buying ΔH_n shares at the end of the n th period $((n-1)\Delta t, n\Delta t] \equiv (t_{n-1}, t_n]$. \tilde{p}_n is determined by the equalisation of supply and demand:

$$f_s(\tilde{p}_n - p_n)\Delta t = f_d(\tilde{p}_n - p_n)\Delta t + \Delta H_n, \quad (1.5.2)$$

An important assumption is the fact that the supply and demand of the small traders are proportional to time and in contrast to the feedback models of section

(1.2) this microeconomic argument gives the *transaction* log price \tilde{p}_n which does not feedback into the price process (p_n) .

We therefore find that the log-price at which the hedger trades is determined by

$$\tilde{p}_n - p_n = \psi\left(\frac{\Delta H_n}{\Delta t}\right), \quad (1.5.3)$$

where ψ is the inverse function to $x \mapsto (f_s(x) - f_d(x))$. We require $\psi(0) = 0$ so that in the absence of the hedger the *transaction* log price equals the notional log price i.e. $\tilde{p}_n = p_n$. If we now let H_t denote the number of shares held by the hedger at time t , and K_t denote the amount of cash held by the hedger at time t , then the notional wealth of the hedger at time t is $w_t \equiv H_t S_t + K_t$. We call, w_t , notional wealth because the hedger cannot transfer $H_t S_t$ to an equivalent amount in cash. The change in notional wealth over the n^{th} period is therefore

$$\begin{aligned} w_{t_n} - w_{t_{n-1}} &= H_{t_{n-1}}(S_{t_n} - S_{t_{n-1}}) + S_{t_n} \Delta H_n - e^{\tilde{p}_n} \Delta H_n \\ &= H_{t_{n-1}}(S_{t_n} - S_{t_{n-1}}) - S_{t_n} h_n (\exp(\psi(h_n)) - 1) \Delta t, \end{aligned} \quad (1.5.4)$$

where we use the notation $h_n = \Delta H_n / \Delta t$. If we let $\Delta t \downarrow 0$, and suppose that H_t is differentiable, with derivative $h_t = dH_t/dt$, then we derive the (continuous-time) dynamics for wealth in the form

$$\begin{aligned} dH_t &= h_t dt, \\ dw_t &= H_t dS_t - h_t S_t f(h_t) dt, \end{aligned} \quad (1.5.5)$$

where $f(x) \equiv \exp(\psi(x)) - 1$ is continuous and increasing, equal to 0 at 0. For small values of x , $f(x) \approx \varepsilon x$ ($\varepsilon > 0$) so we use this form for f for simplicity.¹ in which case the dynamics (1.5.5) become

$$dw_t = H_t dS_t - \varepsilon h_t^2 S_t dt. \quad (1.5.6)$$

¹Economically, there is an unrealistic element with this choice of f . Going back to the discrete time model we have that the average transaction share price is $S_{t_n} e^{\psi(h_n)}$ so that $\Delta K_n = -S_{t_n} e^{\psi(h_n)} \Delta H_n$. $S_{t_n} e^{\psi(h_n)} > 0$ but for our choice of f we have the average transaction share price is $S_{t_n} (1 + \varepsilon h_n)$ which doesn't satisfy this common sense criterion. As an alternative we could take $f(x) \equiv \exp(\varepsilon x) - 1$

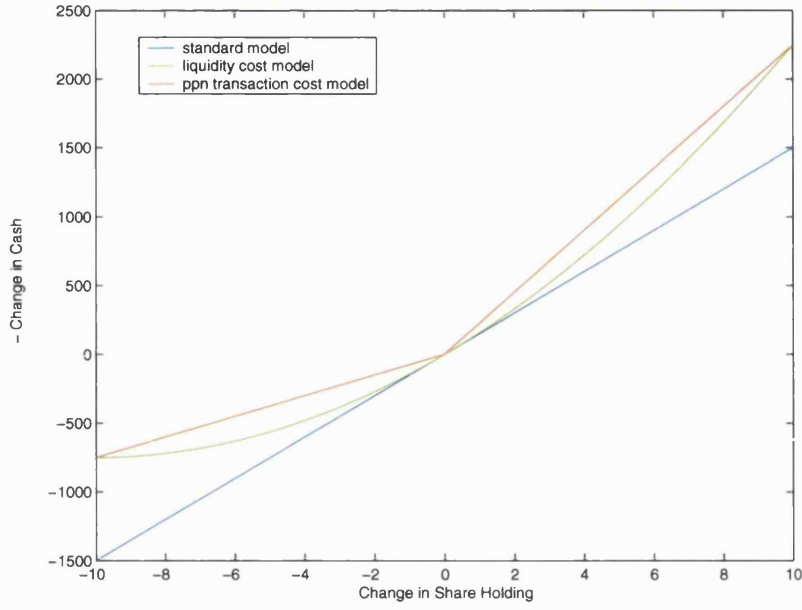


Figure 1-1: Cost

An alternative derivation without making use of a microeconomic argument is to simply modify the notional price, S_{t_n} , to get the transaction price $S_{t_n}e^{\psi(h_n)}$. This is in the spirit of Bakstein and Howison (2003), although their transaction price depends only on ΔH_n and not on Δt .

Above, we plot minus the change in cash, $-\Delta K_n$, as a function of ΔH_n for our choice of $f(x) = \varepsilon x$, for fixed Δt . For the liquidity model we have introduced, $-\Delta K_n = S_{t_n}(1 + \varepsilon \frac{\Delta H_n}{\Delta t})\Delta H_n$. For the standard model² and the proportional transaction cost model we have $-\Delta K_n = S_{t_n}\Delta H_n$ and $-\Delta K_n = S_{t_n}(1 + \lambda 1_{(\Delta H_n \geq 0)} - \mu 1_{(\Delta H_n < 0)})\Delta H_n$ respectively. The parameter values are $S_{t_n} = 150$, $\Delta t = 1$, $\varepsilon = 0.05$ and $\mu = \lambda = 0.5$

²the standard model refers to the case with no transaction costs.

Chapter 2

The Merton Problem in an Illiquid Market

2.1 Review of the Merton problem

In the classical Merton problem of optimal investment and consumption with an infinite horizon an investor may invest in two assets, a cash bond with constant interest rate r , and a share with price process $(S_t)_{t \geq 0}$ satisfying

$$dS_t = S_t(\sigma dW_t + \mu dt) \quad (2.1.1)$$

for constants σ and μ , where $(W_t)_{t \geq 0}$ is a standard Brownian motion.

The investor chooses to consume at a rate C_t and since money can be transferred immediately and costlessly from the risky asset to the cash bond and vice versa we can view the wealth in the risky asset, θ_t , as a control variable instead of the share holding. His wealth evolves as

$$dw_t = (rw_t - C_t)dt + \theta_t(\sigma dW_t + (\mu - r)dt). \quad (2.1.2)$$

Subject to the constraint $w_t \geq 0$ for all t , the investor's objective is to achieve

$$V_M(w) = \sup_{\theta, C} \mathbb{E} \left[\int_0^\infty e^{-\rho t} U(C_t) dt \mid w_0 = w \right] \quad (2.1.3)$$

where ρ is some positive constant and U is a utility function.

(2.1.3) was first solved explicitly in Merton (1969) by finding a solution to the associated Hamilton-Jacobi-Bellman equation. This comes from applying the dynamic programming principle. From the dynamic programming principle we have

$$e^{-\rho t} V_M(w_t) = \sup_{\theta, C} \mathbb{E} \left[\int_t^{t+h} e^{-\rho s} U(C_s) ds + e^{-\rho(t+h)} V_M(w_{t+h}) \middle| \mathcal{F}_t \right] \quad (2.1.4)$$

From Ito's formula applied to $e^{-\rho t} V_M(w_t)$ we have

$$e^{-\rho(t+h)} V_M(w_{t+h}) = e^{-\rho t} V_M(w_t) + \int_t^{t+h} e^{-\rho s} (dV_M(w_s) - \rho V_M(w_s) ds) \quad (2.1.5)$$

Substituting (2.1.5) into (2.1.4) and then assuming the expectation of the stochastic integral with respect to W vanishes we get

$$\sup_{\theta, C} \mathbb{E} \left[\int_t^{t+h} e^{-\rho s} (U(C_s) + (rw_s - C_s + \theta_s(\mu - r))V'_M + \frac{1}{2}\theta_s^2 \sigma^2 V''_M - \rho V_M) ds \right] = 0 \quad (2.1.6)$$

Dividing by h and letting $h \downarrow 0$ we obtain the Hamilton-Jacobi-Bellman equation

$$\sup_{\theta, C} \{ U(C) + (rw - C + \theta(\mu - r))V'_M + \frac{1}{2}\theta^2 \sigma^2 V''_M - \rho V_M \} = 0 \quad (2.1.7)$$

The above derivation is entirely formal. The last technical step once you find a solution to the Hamilton-Jacobi-Bellman equation (HJB) is the “verification”. This is where you show that the solution to the HJB is actually the value function see e.g. Davis and Norman (1990) for details.

If the utility has the form $U(x) = x^{1-R}/(1-R)$ for some $R > 0$ different from

1, Merton finds

$$V_M(w) = \gamma_*^{-R} U(w), \quad (2.1.8)$$

$$\theta_t = \pi_* w_t, \quad (2.1.9)$$

$$C_t = \gamma_* w_t, \quad (2.1.10)$$

where

$$\pi_* = \frac{\mu - r}{\sigma^2 R} \quad (2.1.11)$$

$$\gamma_* = \frac{\rho + (R - 1)(r + (\mu - r)^2 / 2R\sigma^2)}{R} \quad (2.1.12)$$

$$= \frac{\rho + (R - 1)(r + \frac{1}{2}\sigma^2 R\pi_*^2)}{R}. \quad (2.1.13)$$

For a utility of the form $U(x) = \log(x)$ we have

$$V_M(w) = U(w)/\rho + \eta, \quad (2.1.14)$$

$$\theta_t = \pi_* w_t, \quad (2.1.15)$$

$$C_t = \rho w_t, \quad (2.1.16)$$

where

$$\pi_* = \frac{\mu - r}{\sigma^2}, \quad (2.1.17)$$

$$\eta = \frac{\log(\rho)}{\rho} + \frac{(r - \mu)^2 + 2(r - \rho)\sigma^2}{2\rho^2\sigma^2} \quad (2.1.18)$$

2.2 The Merton problem with proportional transaction costs

Using the notation of (1.3.1) the wealth dynamics in the presence of proportional transaction costs are

$$\begin{aligned} d\pi_t^0 &= \pi_t^0(\sigma dW_t + \nu dt) + dL_t - dU_t \\ d\pi_t^1 &= (r\pi_t^1 - C_t)dt - (1 + \lambda)dL_t + (1 - \mu)dU_t \end{aligned} \quad (2.2.1)$$

The objective is

$$V(\pi^0, \pi^1) = \sup \mathbb{E} \left[\int_0^\infty e^{-\rho t} U(C_t) dt \mid \pi_0^0 = \pi^0, \pi_0^1 = \pi^1 \right] \quad (2.2.2)$$

In the classical Merton problem the investor keeps a fixed proportion of wealth in the risky asset. Any attempt to do this here will result in infinite transaction costs. The optimal strategy involves a region where no transactions take place and the buying and selling takes place in order to keep the portfolio within that region. π_t^0 and π_t^1 are constrained to be in the solvency region i.e. if $\pi_t^0 \geq 0$ then $(1 - \mu)\pi_t^0 + \pi_t^1 \geq 0$ and if $\pi_t^1 < 0$ then $(1 + \lambda)\pi_t^0 + \pi_t^1 \geq 0$. An important property of the value function is the homothetic property i.e. for $\lambda > 0$

$$V(\lambda \pi^0, \lambda \pi^1) = \lambda^{1-R} V(\pi^0, \pi^1) \quad (2.2.3)$$

Following Davis and Norman (1990) to get an idea of the solution we restrict trading strategies for L_t and U_t to ones of the form $L_t = \int_0^t l_s ds$ and $U_t = \int_0^t u_s ds$ where $0 \leq l_s, u_s \leq k$ for some constant k . The Hamilton-Jacobi-Bellman equation for this problem is

$$\sup_{C, l, u} \left\{ \frac{1}{2} \sigma^2 (\pi^0)^2 V_{\pi^0 \pi^0} + V_{\pi^0} (\pi^0 \nu + l - u) + V_{\pi^1} (r \pi^1 - C - (1 + \lambda)l + (1 - \mu)u) + \frac{C^{1-R}}{1-R} - \rho V \right\} = 0 \quad (2.2.4)$$

The maximization over C, l and u yields:

$$C^{-R} = V_{\pi^1}$$

$$l = \begin{cases} k & \text{if } V_{\pi^0} \geq V_{\pi^1}(1 + \lambda), \\ 0 & \text{if } V_{\pi^0} < V_{\pi^1}(1 + \lambda) \end{cases}$$

$$u = \begin{cases} k & \text{if } V_{\pi^0} \leq V_{\pi^1}(1 - \mu), \\ 0 & \text{if } V_{\pi^0} > V_{\pi^1}(1 - \mu) \end{cases} \quad (2.2.5)$$

The optimal control is therefore of bang-bang type. An investor trades at the

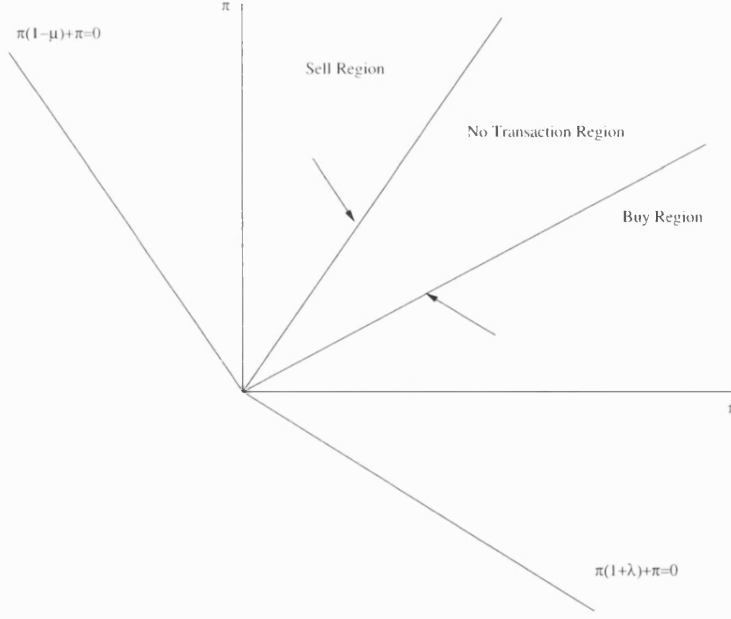


Figure 2-1: Transaction regions

optimal rate or not at all. If we now use the homothetic property of the original problem we find that

$$\begin{aligned} V_{\pi^0}(\lambda\pi^0, \lambda\pi^1) &= \lambda^{-R}V_{\pi^0}(\pi^0, \pi^1) \text{ and} \\ V_{\pi^1}(\lambda\pi^0, \lambda\pi^1) &= \lambda^{-R}V_{\pi^1}(\pi^0, \pi^1) \end{aligned} \quad (2.2.6)$$

so that if we plot π^0 against π^1 then along a line through the origin $\frac{V_{\pi^1}}{V_{\pi^0}}$ is a constant which suggests the boundaries of the no-transaction region are straight lines through the origin. Davis and Norman (1990) prove this rigorously. There is no known closed form solution for the boundaries and value function. They have to be computed numerically.

2.3 The Merton problem in an illiquid market

Using (1.5.6) we have the following equations for the evolution of the asset and the wealth of the investor:

$$dS_t = S_t(\mu dt + \sigma dW_t), \quad (2.3.1)$$

$$dw_t = rw_t dt + H_t(dS_t - S_t r dt) - C_t dt - \varepsilon h_t^2 S_t dt. \quad (2.3.2)$$

Here the situation is different from the classical Merton problem in two ways. Not only do we have an extra term due to the cost of liquidity but we no longer control the share holding, or if you prefer the wealth in the risky asset. Consider the situation in the classical Merton problem with $0 < \pi_* < 1$, you buy and sell shares to maintain your wealth partitioned according to the optimal ratio π_* . In the illiquid case, however, because $H_t = \int_0^t h_u du$ you cannot adjust your portfolio quickly enough to keep it at the optimal ratio without incurring an infinite liquidity cost, so H_t is no longer a function of the underlying Brownian motion. The control variables for this problem are C_t and h_t with the objective:

$$V(w, H, S) = \sup \mathbb{E} \left[\int_0^\infty e^{-\rho t} U(C_t) dt \mid w_0 = w, H_0 = H, S_0 = S \right], \quad (2.3.3)$$

with the restriction $w_t \geq 0$ for all t .

In the classical Merton problem it is permissible to borrow to invest in the risky asset and to sell the risky asset short and invest more than all your wealth in the cash bond. In this illiquid market, however, we find that the non-negative wealth constraint implies that we cannot sell the risky asset short nor borrow to invest in the risky asset.

Proposition 1. *If $w_t = H_t S_t + K_t \geq 0$ and $K_t < 0$ then $\mathbb{P}(w_s \geq 0; \forall s > t) < 1$*

Proof See Appendix. ■

Proposition 2. *If $w_t = H_t S_t + K_t \geq 0$ and $H_t < 0$ then $\mathbb{P}(w_s \geq 0; \forall s > t) < 1$*

Proof See Appendix. ■

Longstaff (2001) has the same conclusion for an illiquid market with a bounded h_t . The relaxed investor of Rogers (2001) who reviews his portfolio and consumption at fixed intervals also cannot sell the risky asset short nor borrow to invest in the risky asset. Combining the constraints $H_t \geq 0$ and $K_t \geq 0$ we have

$$\frac{H_t S_t}{w_t} \in [0, 1] \quad (2.3.4)$$

In other words the proportion of wealth in the risky asset is in $[0, 1]$.

Given initial values for H , w and S , we say that (C, h) is *admissible* if (2.3.2) has a unique strong solution and $w_t \geq 0$ for all t .

For $U(x) = x^{1-R}/(1-R)$ we can exploit the scaling in (2.3.2), (2.3.1) and (2.3.3) to give

$$\lambda^{1-R} V(w, H, S) = V(\lambda w, H, \lambda S), \quad (2.3.5)$$

$$\lambda C^*(w, H, S) = C^*(\lambda w, H, \lambda S) \quad (2.3.6)$$

Let $\lambda = 1/S$.

$$V(w, H, S) = V(w/S, H, 1) S^{1-R} = v(z, H) S^{1-R}, \quad (2.3.7)$$

$$C^*(w, H, S) = C^*(w/S, H, 1) S = c^*(z, H) S \quad (2.3.8)$$

where $v(z, H) = V(z, H, 1)$, $c^*(z, H) = C^*(z, H, 1)$, $z = w/S$, and $C^*(w, H, S)$ is the optimal consumption rate as a function of current wealth, current holding of the share, and current share price. To see this note that a policy (C, h) is admissible for starting values H_0 , w_0 and S_0 if and only if $(\lambda C, h)$ is admissible for starting values H_0 , λw_0 and λS_0 where $\lambda > 0$.

Substituting (2.3.7) and (2.3.8) into the objective (2.3.3) we have

$$v(z, H) = \sup \mathbb{E} \left[\int_0^\infty e^{-\rho t} U(c^*(z_t, H_t)) \left(\frac{S_t}{S_0} \right)^{1-R} dt \middle| z_0 = z, H_0 = H \right] \quad (2.3.9)$$

The left-hand side of (2.3.9) is a function of z and H only where-as the right-hand side also involves the share price S_t but we can remove this dependence on the share price by changing measure.

By applying Ito's formula we find that under the original measure \mathbb{P} , $z_t = \frac{w_t}{s_t}$ satisfies

$$dz_t = (H_t - z_t)\sigma dW_t + ((\mu - r - \sigma^2)(H_t - z_t) - c_t - \epsilon h_t^2)dt \quad (2.3.10)$$

and

$$\left(\frac{S_t}{S_0} \right)^{1-R} = \exp((1-R)\sigma W_t + (1-R)(\mu - \frac{1}{2}\sigma^2)t) \quad (2.3.11)$$

We define a new measure by

$$\left. \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \right|_{\mathcal{F}_t} = \exp((1-R)\sigma W_t - \frac{1}{2}(1-R)^2\sigma^2 t) \quad (2.3.12)$$

Under the new measure $\tilde{\mathbb{P}}$

$$\tilde{W}_t = W_t - (1-R)\sigma t \quad (2.3.13)$$

is Brownian motion and

$$v(z, H) = \sup \mathbb{E}_{\tilde{\mathbb{P}}} \left[\int_0^\infty e^{-\tilde{\rho} t} \frac{c_t^{1-R}}{1-R} dt \middle| z_0 = z, H_0 = H \right] \quad (2.3.14)$$

and

$$dz_t = (H_t - z_t)\sigma d\tilde{W}_t + ((\mu - r - R\sigma^2)(H_t - z_t) - c_t - \epsilon h_t^2)dt \quad (2.3.15)$$

where $\tilde{\rho} = \rho - (1-R)(\mu - \frac{1}{2}R\sigma^2)$.

We are now in a position to state the Hamilton-Jacobi-Bellman (HJB) equation for this problem.

$$\sup_{c,h} \left\{ c^{1-R}/(1-R) - \tilde{\rho}v + \frac{1}{2}\sigma^2(z-H)^2v_{zz} - (\varepsilon h^2 + c + \alpha(z-H))v_z + hv_H \right\} = 0 \quad (2.3.16)$$

where $\alpha = \mu - r - \sigma^2 R$.

The maximisation over c and h gives

$$c^* = v_Z^{-1/R} \quad (2.3.17)$$

$$h^* = \frac{v_H}{2\varepsilon v_Z}. \quad (2.3.18)$$

We can proceed in a similar way to derive the HJB equation for $U(x) = \log(x)$. The scaling in (2.3.2), (2.3.1) and (2.3.3) gives for $\lambda > 0$

$$V(w, H, S) + \frac{\log(\lambda)}{\rho} = V(\lambda w, H, \lambda S) \quad (2.3.19)$$

$$\lambda C^*(w, H, S) = C^*(\lambda w, H, \lambda S) \quad (2.3.20)$$

Let $\lambda = 1/S$

$$V(w, H, S) = V(w/S, H, 1) + \frac{\log(S)}{\rho} = v(z, H) + \frac{\log(S)}{\rho} \quad (2.3.21)$$

$$C^*(w, H, S) = C^*(w/S, H, 1)S = c^*(z, H)S \quad (2.3.22)$$

and

$$\begin{aligned} v(z, H) + \frac{\log(S)}{\rho} &= \mathbb{E} \left[\int_0^\infty e^{-\rho t} \log(c(z_t, H_t)) dt \middle| z_0 = z, H_0 = H \right] \\ &+ \mathbb{E} \left[\int_0^\infty e^{-\rho t} \log(S_t) dt \middle| S_0 = S \right] \end{aligned} \quad (2.3.23)$$

so we have

$$v(z, H) = \mathbb{E} \left[\int_0^\infty e^{-\rho t} \log(c(z_t, H_t)) dt \middle| z_0 = z, H_0 = H \right] + \frac{\mu - \frac{1}{2}\sigma^2}{\rho^2}$$

Then the HJB equation is

$$\sup_{c,h} \left\{ \log(c) - \rho F + \frac{1}{2} \sigma^2 (z - H)^2 F_{zz} - (\varepsilon h^2 + c + \alpha(z - H)) F_z + h F_H \right\} = 0 \quad (2.3.24)$$

where $F(z, H) = v(z, H) - \frac{\mu - \frac{1}{2}\sigma^2}{\rho^2}$

The maximisation over c and h yields

$$c^* = F_z^{-1} \quad (2.3.25)$$

$$h^* = \frac{F_H}{2\varepsilon F_z}. \quad (2.3.26)$$

2.4 Numerical Procedure

In this section we numerically solve the problem (2.3.14) with a power utility function i.e. $U(x) = x^{1-R}/(1-R)$ and $0 < R < 1$.

Firstly we make a change of variable from $(H, z)'$ to $(H, Y)'$ where $Y = z - H$ i.e. Y is the cash holding measured in units of the share. We do this because we have a constraint $0 \leq H \leq z$ which is far better expressed as $H \geq 0$ and $Y \geq 0$ when using a grid for numerical work.

The Hamilton-Jacobi-Bellman equation for this problem (2.3.16) is transformed to

$$\sup_{c,h} \left\{ c^{1-R}/(1-R) - \tilde{\rho}v + \frac{1}{2} \sigma^2 Y^2 v_{YY} - (h + \varepsilon h^2 + c + \alpha Y) v_Y + h v_H \right\} = 0 \quad (2.4.1)$$

where $\alpha = \mu - r - \sigma^2 R$, $\tilde{\rho} = \rho + (R - 1)(\mu - \frac{1}{2}\sigma^2 R)$.

and

$$\begin{aligned} c^* &= v_Y^{-1/R} \\ h^* &= \frac{v_H - v_Y}{2\varepsilon v_Y} \end{aligned} \quad (2.4.2)$$

Infinite horizon control problems are notoriously difficult to solve numerically. The HJB (2.4.1) together with the controls in feedback form (2.4.2) suggests a possible technique. If we fix initial guesses for the controls, c and h , as a function of the state space then

$$c^{1-R}/(1-R) - \tilde{\rho}v + \frac{1}{2}\sigma^2 Y^2 v_{YY} - (h + \varepsilon h^2 + c + \alpha Y)v_Y + hv_H = 0 \quad (2.4.3)$$

is a linear pde and we can solve for v using a finite difference scheme for the derivatives. Using this v we can update the controls using the right hand side of (2.4.2). With these new controls we substitute them into (2.4.3) and solve for a new v and update the controls again and so on. Unfortunately there is no theoretical justification for the convergence of this procedure to the value function in (2.3.14) and indeed trying this we find we do not get any convergence. We now use a Markov chain approximation method to find an approximate solution for the value function. It is based on the technique described in Kushner and Dupuis (1992).

2.4.1 Markov Chain Approximation

Let $d > 0$ be a scalar approximation parameter (the grid spacing goes to zero as $d \downarrow 0$). We approximate the controlled state variable process $\begin{pmatrix} Y_t \\ H_t \end{pmatrix}_{t \geq 0}$ with a controlled discrete parameter markov chain $\xi = (\xi_n)_{n \in \mathbb{N}}$ on a finite discrete state space $S = \left\{ \begin{pmatrix} u_i \\ v_j \end{pmatrix} : i \in \{1, 2, \dots, n\}, j \in \{1, 2, \dots, m\} \right\}$ where, at the point with co-ordinates (i, j) , u_i is the cash amount in units of the share and v_j is the number of shares held. The discrete state space S is a rectangular grid with equal spacings i.e. $u_{i+1} - u_i = \Delta u$, $i \in \{1, 2, \dots, n-1\}$ and $v_{j+1} - v_j = \Delta v$, $j \in \{1, 2, \dots, m-1\}$. Define $S_0 = \left\{ \begin{pmatrix} u_i \\ v_j \end{pmatrix} : i \in \{2, \dots, n-1\}, j \in \{2, \dots, m-1\} \right\}$ and $\partial S = S \setminus S_0$. The evolution of the markov chain is governed by transition probabilities $p(x, y|h, c)$ denoting the probability of the markov chain going from $x \in S$ to $y \in S$ when

the control (h, c) is applied. Let (h_n, c_n) denote the control applied for the chain at time n .

In our problem the diffusion is described by

$$\begin{pmatrix} dY_t \\ dH_t \end{pmatrix} = \begin{pmatrix} -\sigma Y_t \\ 0 \end{pmatrix} dW_t + \begin{pmatrix} -\alpha Y_t - (h_t + \varepsilon h_t^2 + c_t) \\ h_t \end{pmatrix} dt \quad (2.4.4)$$

Let us define an “interpolation interval” $\Delta t_n = \Delta t(\xi_n, h_n, c_n)$ which is the time interval until the next the jump for ξ at time n . Define the difference, $\Delta \xi_n = \xi_{n+1} - \xi_n$, and the conditional mean and variance of the chain as

$$\begin{aligned} \mu_n &= \mu(x, h, c) = E[\Delta \xi_n | \xi_n = x, h_n = h, c_n = c] \\ \Sigma_n &= \Sigma(x, h, c) = E[(\Delta \xi_n - \mu_n)(\Delta \xi_n - \mu_n)^\top | \xi_n = x, h_n = h, c_n = c] \end{aligned}$$

The approximating markov chain obeys “local consistency conditions” for the markov chain solution V^d to converge to the continuous time solution as $d \downarrow 0$.

We require that

$$\begin{aligned} \mu(x, h, c) &= \begin{pmatrix} -\alpha u_i - (h + \varepsilon h^2 + c) \\ h \end{pmatrix} \Delta t(x, h, c) + o(\Delta t(x, h, c)) \\ \Sigma(x, h, c) &= \begin{pmatrix} (\sigma u_i)^2 & 0 \\ 0 & 0 \end{pmatrix} \Delta t(x, h, c) + o(\Delta t(x, h, c)) \\ \sup_{n, \omega} |\xi_{n+1} - \xi_n| &\xrightarrow{d} 0 \end{aligned}$$

We approximate the continuous time objective (2.3.14) with the discrete time objective

$$V^d(x) = \sup_{(h(\cdot), c(\cdot))} \mathbb{E} \left[\sum_{n=0}^{\infty} e^{-\tilde{\rho} t_n} U(c(\xi_n)) \Delta t_n \mid \xi_0 = x \in S \right]$$

where $t_n = \sum_{k=0}^{n-1} \Delta t_k$ and $t_0 = 0$.

The dynamic programming equation for this problem is

$$V^d(x) = \sup_{(h,c)} \left\{ U(c) \Delta t(x, h, c) + e^{-\bar{\rho} \Delta t(x, h, c)} \sum_{y \in S} p(x, y | h, c) V^d(y) \right\} \quad (2.4.5)$$

We use a policy improvement algorithm to find $V^d(x)$. This works as follows:

Given a control $(h_0(\cdot), c_0(\cdot))$ we calculate

$$V_0^d(x) = \left\{ U(c_0(x)) \Delta t(x, h_0(x), c_0(x)) + e^{-\bar{\rho} \Delta t(x, h_0(x), c_0(x))} \sum_{y \in S} p(x, y | h_0(x), c_0(x)) V_0^d(y) \right\}, \quad \forall x \in S \quad (2.4.6)$$

This is done using the sparse linear system solver `lusolve` in `scilab`.

Then we calculate a new control $(h_1(\cdot), c_1(\cdot))$ as

$$(h_1(x), c_1(x)) = \arg \max_{(h,c)} \left\{ U(c) \Delta t(x, h, c) + e^{-\bar{\rho} \Delta t(x, h, c)} \sum_{y \in S} p(x, y | h, c) V_0^d(y) \right\}, \quad \forall x \in S$$

Continuing this way we have “policy evaluations”

$$V_k^d(x) = \left\{ U(c_k(x)) \Delta t(x, h_k(x), c_k(x)) + e^{-\bar{\rho} \Delta t(x, h_k(x), c_k(x))} \sum_{y \in S} p(x, y | h_k(x), c_k(x)) V_k^d(y) \right\}, \quad \forall x \in S \quad (2.4.7)$$

and “policy improvements”

$$(h_{k+1}(x), c_{k+1}(x)) = \arg \max_{(h,c)} \Psi_k(x, h, c) \quad \forall x \in S \quad (2.4.8)$$

where $\Psi_k(x, h, c) = U(c) \Delta t(x, h, c) + e^{-\bar{\rho} \Delta t(x, h, c)} \sum_{y \in S} p(x, y | h, c) V_k^d(y)$.

and $V_k^d(x) \rightarrow V^d(x)$.

2.4.2 Transition Probabilities

An upwind finite difference scheme offers a way to get locally consistent transition probabilities. We use an upwind scheme since a central finite difference scheme will not give positive transition probabilities. Define $b(x, h(x), c(x)) = -(h(x) + \varepsilon h^2(x) + c(x) + \alpha u_i)$ and the upwind differential operators

$$\begin{aligned} D_H^+ V^d(u_i, v_j) &= \frac{V^d(u_i, v_j + \Delta v) - V^d(u_i, v_j)}{\Delta v} \\ D_H^- V^d(u_i, v_j) &= \frac{V^d(u_i, v_j) - V^d(u_i, v_j - \Delta v)}{\Delta v} \\ D_Y^+ V^d(u_i, v_j) &= \frac{V^d(u_i + \Delta u, v_j) - V^d(u_i, v_j)}{\Delta u} \\ D_Y^- V^d(u_i, v_j) &= \frac{V^d(u_i, v_j) - V^d(u_i - \Delta u, v_j)}{\Delta u} \end{aligned}$$

The approximations we use for the first derivatives are

$$h(x)V_H \approx h(x)^+ D_H^+ V^d(u_i, v_j) - h(x)^- D_H^- V^d(u_i, v_j) \quad (2.4.9)$$

$$b(x, h(x), c(x))V_Y \approx b(x, h(x), c(x))^+ D_Y^+ V^d(u_i, v_j) - b(x, h(x), c(x))^- D_Y^- V^d(u_i, v_j)$$

and for the second derivative

$$V_{YY} \approx \frac{V^d(u_i + \Delta u, v_j) - 2V^d(u_i, v_j) + V^d(u_i - \Delta u, v_j)}{(\Delta u)^2} \quad (2.4.10)$$

The generator for the (2.4.4) is $\mathcal{L}V = \frac{1}{2}\sigma^2 Y^2 V_{YY} - (\alpha Y + h + \varepsilon h^2 + c)V_Y + hV_H$. To find transition probabilities we substitute the approximations (2.4.9) and

(2.4.10) into $\mathcal{L}V(x) = 0$ and make $V^d(u_i, v_j)$ the subject. This gives

$$\begin{aligned}
V^d(u_i, v_j) = & \left(\frac{b(x, h(x), c(x))^-}{\Delta u} + \frac{\kappa(x)}{(\Delta u)^2} \right) / Q(x, h(x), c(x)) V^d(u_i - \Delta u, v_j) \\
& + \left(\frac{b(x, h(x), c(x))^+}{\Delta u} + \frac{\kappa(x)}{(\Delta u)^2} \right) / Q(x, h(x), c(x)) V^d(u_i + \Delta u, v_j) \\
& + \frac{h(x)^-}{\Delta v} / Q(x, h(x), c(x)) V^d(u_i, v_j - \Delta v) \\
& + \frac{h(x)^+}{\Delta v} / Q(x, h(x), c(x)) V^d(u_i, v_j + \Delta v)
\end{aligned} \tag{2.4.11}$$

where $\kappa(x) = \frac{1}{2}(u_i \sigma)^2$ and $Q(x, h(x), c(x)) = \frac{|b(x, h(x), c(x))|}{\Delta u} + \frac{|h(x)|}{\Delta v} + \frac{2\kappa(x)}{(\Delta u)^2}$. Interpreting the coefficients of V^d 's on the right hand side as transition probabilities gives

$$\begin{aligned}
p\left(\begin{pmatrix} u_i \\ v_j \end{pmatrix}, \begin{pmatrix} u_i - \Delta u \\ v_j \end{pmatrix}\right) &= \left(\frac{b(x, h(x), c(x))^-}{\Delta u} + \frac{\kappa(x)}{(\Delta u)^2} \right) / Q(x, h(x), c(x)) \\
p\left(\begin{pmatrix} u_i \\ v_j \end{pmatrix}, \begin{pmatrix} u_i + \Delta u \\ v_j \end{pmatrix}\right) &= \left(\frac{b(x, h(x), c(x))^+}{\Delta u} + \frac{\kappa(x)}{(\Delta u)^2} \right) / Q(x, h(x), c(x)) \\
p\left(\begin{pmatrix} u_i \\ v_j \end{pmatrix}, \begin{pmatrix} u_i \\ v_j - \Delta v \end{pmatrix}\right) &= \frac{h(x)^-}{\Delta v} / Q(x, h(x), c(x)) \\
p\left(\begin{pmatrix} u_i \\ v_j \end{pmatrix}, \begin{pmatrix} u_i \\ v_j + \Delta v \end{pmatrix}\right) &= \frac{h(x)^+}{\Delta v} / Q(x, h(x), c(x))
\end{aligned} \tag{2.4.12}$$

for $x \in S_0$ and $\Delta t(x, h, c) = 1/Q(x, h, c)$.

We are going to use a policy improvement algorithm so ideally we would like control independent denominators for the transition probabilities of (2.4.12) so that the policy improvement step (2 – 13) can be done as simply as possible. For the numerical procedure the controls are bounded, $0 \leq c(x) \leq cu$ and $-hu = hl \leq h(x) \leq hu$ so we can amend the transition probabilities but still have them locally consistent with the diffusion by allowing transitions of the states of the chain to themselves. The new transition probabilities for $x \in S_0$ are:

$$\begin{aligned}
p\left(\begin{pmatrix} u_i \\ v_j \end{pmatrix}, \begin{pmatrix} u_i - \Delta u \\ v_j \end{pmatrix}\right) &= \left(\frac{b(x, h(x), c(x))^-}{\Delta u} + \frac{\kappa(x)}{(\Delta u)^2}\right)/Q(x) \\
p\left(\begin{pmatrix} u_i \\ v_j \end{pmatrix}, \begin{pmatrix} u_i + \Delta u \\ v_j \end{pmatrix}\right) &= \left(\frac{b(x, h(x), c(x))^+}{\Delta u} + \frac{\kappa(x)}{(\Delta u)^2}\right)/Q(x) \\
p\left(\begin{pmatrix} u_i \\ v_j \end{pmatrix}, \begin{pmatrix} u_i \\ v_j - \Delta v \end{pmatrix}\right) &= \frac{h(x)^-}{\Delta v}/Q(x) \\
p\left(\begin{pmatrix} u_i \\ v_j \end{pmatrix}, \begin{pmatrix} u_i \\ v_j + \Delta v \end{pmatrix}\right) &= \frac{h(x)^+}{\Delta v}/Q(x) \\
p\left(\begin{pmatrix} u_i \\ v_j \end{pmatrix}, \begin{pmatrix} u_i \\ v_j \end{pmatrix}\right) &= \left(Q(x) - \left(\frac{|b(x, h(x), c(x))|}{\Delta u} + \frac{|h(x)|}{\Delta v} + \frac{2\kappa(x)}{(\Delta u)^2}\right)\right)/Q(x)
\end{aligned} \tag{2.4.13}$$

$Q(x) = \frac{M(x)}{\Delta u} + \frac{hu}{\Delta v} + \frac{2\kappa(x)}{(\Delta u)^2}$ where $M(x) = \max(-\alpha u_i + \frac{1}{4\varepsilon}, cu + \varepsilon(hu)^2 + hu)$ and $\Delta t(x) = 1/Q(x)$.

Now we check that (2.4.13) is in fact a locally consistent set of transition probabilities.

$$\begin{aligned}
\mathbb{E}[\Delta \xi_n | \xi_n = x] &= \begin{pmatrix} \Delta u \\ 0 \end{pmatrix} \left(\frac{b(x, h(x), c(x))^+}{\Delta u} + \frac{\kappa(x)}{(\Delta u)^2}\right)/Q(x) \\
&+ \begin{pmatrix} -\Delta u \\ 0 \end{pmatrix} \left(\frac{b(x, h(x), c(x))^-}{\Delta u} + \frac{\kappa(x)}{(\Delta u)^2}\right)/Q(x) \\
&+ \begin{pmatrix} 0 \\ \Delta v \end{pmatrix} \frac{h(x)^+}{\Delta v}/Q(x) \\
&+ \begin{pmatrix} 0 \\ -\Delta v \end{pmatrix} \frac{h(x)^-}{\Delta v}/Q(x) \\
&= \begin{pmatrix} b(x, h(x), c(x)) \\ h(x) \end{pmatrix} \Delta t_n
\end{aligned} \tag{2.4.14}$$

and

$$\begin{aligned}
& \mathbb{E}[(\Delta\xi_n - \mathbb{E}[\Delta\xi_n])(\Delta\xi_n - \mathbb{E}[\Delta\xi_n])^\top | \xi_n = x] = \\
& \begin{pmatrix} (\Delta u - b(x, h(x), c(x))\Delta t_n)^2 & -h(x)\Delta t_n(\Delta u - b(x, h(x), c(x))\Delta t_n) \\ -h(x)\Delta t_n(\Delta u - b(x, h(x), c(x))\Delta t_n) & (h(x)\Delta t_n)^2 \end{pmatrix} \left(\frac{(b(x, h(x), c(x))^+}{\Delta u} + \frac{\kappa(x)}{(\Delta u)^2} \right) / Q(x) + \\
& \begin{pmatrix} (\Delta u + b(x, h(x), c(x))\Delta t_n)^2 & h(x)\Delta t_n(\Delta u + b(x, h(x), c(x))\Delta t_n) \\ h(x)\Delta t_n(\Delta u + b(x, h(x), c(x))\Delta t_n) & (h(x)\Delta t_n)^2 \end{pmatrix} \left(\frac{(b(x, h(x), c(x))^-}{\Delta u} + \frac{\kappa(x)}{(\Delta u)^2} \right) / Q(x) + \\
& \begin{pmatrix} (b(x, h(x), c(x))\Delta t_n)^2 & -b(x, h(x), c(x))\Delta t_n(\Delta v - h(x)\Delta t_n) \\ -b(x, h(x), c(x))\Delta t_n(\Delta v - h(x)\Delta t_n) & (\Delta v - h(x)\Delta t_n)^2 \end{pmatrix} \frac{h(x)^+}{\Delta v} / Q(x) + \\
& \begin{pmatrix} (b(x, h(x), c(x))\Delta t_n)^2 & b(x, h(x), c(x))\Delta t_n(\Delta v + h(x)\Delta t_n) \\ b(x, h(x), c(x))\Delta t_n(\Delta v + h(x)\Delta t_n) & (\Delta v + h(x)\Delta t_n)^2 \end{pmatrix} \frac{h(x)^-}{\Delta v} / Q(x) + \\
& \begin{pmatrix} (b(x, h(x), c(x))\Delta t_n)^2 & b(x, h(x), c(x))h(x)(\Delta t_n)^2 \\ b(x, h(x), c(x))h(x)(\Delta t_n)^2 & (h(x)\Delta t_n)^2 \end{pmatrix} \left(1 - \left(\frac{|b(x, h(x), c(x))|}{\Delta u} + \frac{|h|}{\Delta v} + \frac{2\kappa(x)}{(\Delta u)^2} \right) / Q(x) \right) \\
& = \begin{pmatrix} 2\kappa(x) & 0 \\ 0 & 0 \end{pmatrix} \Delta t_n + o(\Delta t_n)
\end{aligned} \tag{2.4.15}$$

so (2.4.13) is acceptable.

2.4.3 Policy Improvement Algorithm

Due to upwind finite differencing, if we are given $V_k^d(\cdot)$ it is not a closed form calculation to do the policy improvement step (2-13). We have to take into account the signs of $h(\cdot)$ and $b(\cdot, h(\cdot), c(\cdot))$.

For $x \in S_0$ have 4 cases to consider for the optimal control $(h_{k+1}(x), c_{k+1}(x))$.

Case 1.

Assume $h_{k+1}(x) \geq 0$ and $b(x, h_{k+1}(x), c_{k+1}(x)) \geq 0$

$$\begin{aligned}
h^* &= \max(\min(hu, (\frac{D_H^+ V_k^d(u_i, v_j)}{D_Y^+ V_k^d(u_i, v_j)} - 1)/(2\varepsilon)), hl) \text{ and} \\
c^* &= \min(cu, (e^{-\tilde{\rho}/Q(x)} D_Y^+ V_k^d(u_i, v_j))^{-1/R})
\end{aligned}$$

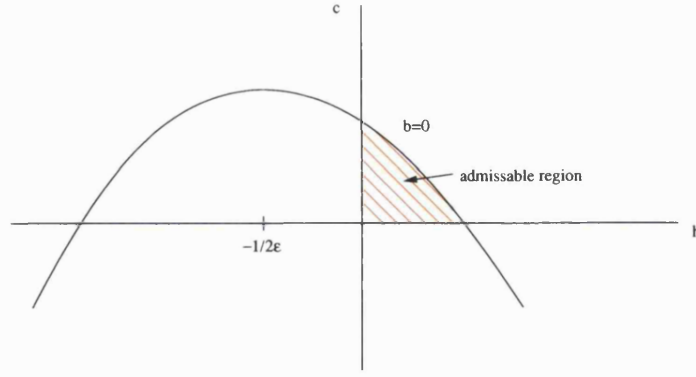


Figure 2-2: Case 1

if $h^* \geq 0$ and $b(x, h^*, c^*) \geq 0$ then $h1(x) = h^*$ and $c1(x) = c^*$

else

if $h^* \leq 0$ then

$$h1(x) = 0 \text{ and } c1(x) = \min(c^*, -\alpha u_i)$$

else

$$(h1(x), c1(x)) = \arg \max_{cu \geq c \geq 0, hu \geq h \geq 0, b(x, h, c) = 0} \left\{ h e^{-\tilde{\rho}/Q(x)} D_H^+ V_k^d(u_i, v_j) + \frac{c^{1-R}}{1-R} \right\}$$

Case 2.

Assume $h_{k+1}(x) \geq 0$ and $b(x, h_{k+1}(x), c_{k+1}(x)) \leq 0$

$$h^* = \max(\min(hu, (\frac{D_H^+ V_k^d(u_i, v_j)}{D_Y^- V_k^d(u_i, v_j)} - 1)/(2\varepsilon)), hl), \text{ and}$$

$$c^* = \min(cu, (e^{-\tilde{\rho}/Q(x)} D_Y^- V_k^d(u_i, v_j))^{-1/R})$$

if $h^* \geq 0$ and $b(x, h^*, c^*) \leq 0$ then $h2(x) = h^*$ and $c2(x) = c^*$

else

if $c^* \geq -\alpha u_i$ then

$$h2(x) = 0 \text{ and } c2(x) = c^*$$

else

$$(h2(x), c2(x)) = \arg \max_{cu \geq c \geq 0, hu \geq h \geq 0, b(x, h, c) = 0} \left\{ h e^{-\tilde{\rho}/Q(x)} D_H^+ V_k^d(u_i, v_j) + \frac{c^{1-R}}{1-R} \right\}$$

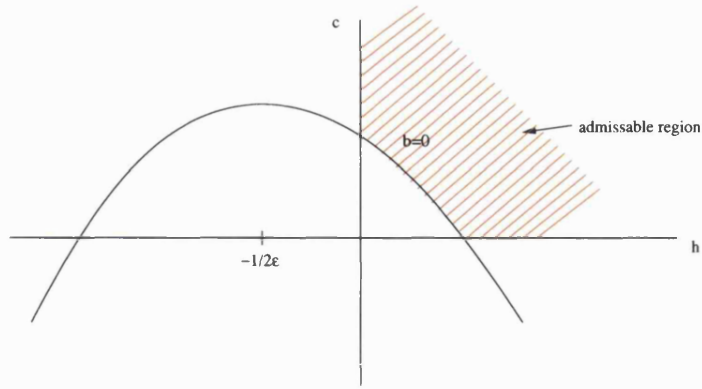


Figure 2-3: Case 2

Case 3.

Assume $h_{k+1}(x) \leq 0$ and $b(x, h_{k+1}(x), c_{k+1}(x)) \geq 0$

$$h^* = \max(\min(hu, (\frac{D_H^- V_k^d(u_i, v_j)}{D_Y^+ V_k^d(u_i, v_j)} - 1)/(2\varepsilon))hl), \text{ and}$$

$$c^* = \min(cu, (e^{-\tilde{\rho}/Q(x)} D_Y^+ V_k^d(u_i, v_j))^{-1/R})$$

if $h^* \leq 0$ and $b(x, h^*, c^*) \geq 0$ then $h3(x) = h^*$ and $c3(x) = c^*$

else

if $c^* \leq -\alpha u_i$ and $h^* \geq 0$ then

$$h3(x) = 0 \text{ and } c3(x) = c^*$$

else

$$(h3(x), c3(x)) = \arg \max_{cu \geq c \geq 0, hl \leq h \leq 0, b(x, h, c) = 0} \left\{ h e^{-\tilde{\rho}/Q(x)} D_H^- V_k^d(u_i, v_j) + \frac{c^{1-R}}{1-R} \right\}$$

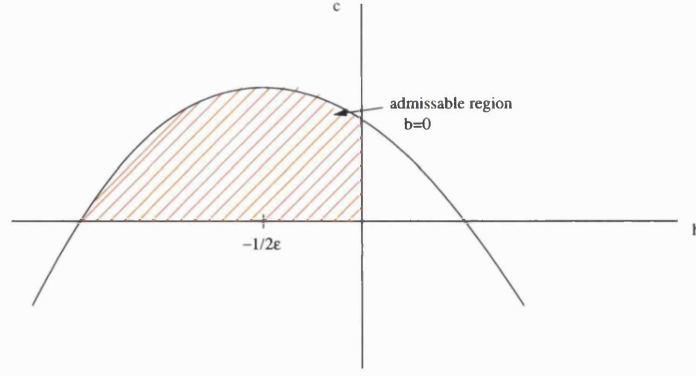


Figure 2-4: Case 3

Case 4.

Assume $h_{k+1}(x) \leq 0$ and $b(x, h_{k+1}(x), c_{k+1}(x)) \leq 0$

$$h^* = \max(\min(hu, (\frac{D_H^- V^d(u_i, v_j)}{D_Y^- V_k^d(u_i, v_j)} - 1)/(2\varepsilon)), hl), \text{ and}$$

$$c^* = \min(cu, (e^{-\tilde{\rho}/Q(x)} D_Y^- V_k^d(u_i, v_j))^{-1/R})$$

if $h^* \leq 0$ and $b(x, h^*, c^*) \leq 0$ then $h4(x) = h^*$ and $c4(x) = c^*$
else

if $h^* \geq 0$ and $c^* \geq -\alpha u_i$ then

$$h4(x) = 0 \text{ and } c4(x) = c^*$$

else

$$(h4(x), c4(x)) = \arg \max_{cu \geq c \geq 0, hl \leq h \leq 0, b(x, h, c) = 0} \left\{ h e^{-\tilde{\rho}/Q(x)} D_H^- V_k^d(u_i, v_j) + \frac{c^{1-R}}{1-R} \right\}$$

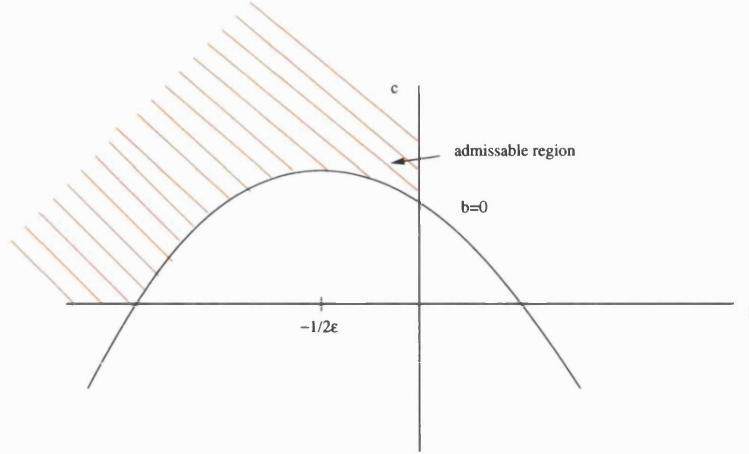


Figure 2-5: Case 4

and finally

$$(h_{k+1}(x), c_{k+1}(x)) = \arg \max_{(h,c) \in \{(h_1(x), c_1(x)), (h_2(x), c_2(x)), (h_3(x), c_3(x)), (h_4(x), c_4(x))\}} \Psi(x, h, c)$$

On the boundary $x \in \partial S$

Given that we have the constraints $H \geq 0$ and $Y \geq 0$ for admissible strategies this takes care of the boundary conditions for $v_j = 0$ and $u_i = 0$. For $v_j = H_{max}$ (the largest value for v_j on the grid) we add the constraint $h(x) \leq 0$ and for $u_i = Y_{max}$ (the largest value of u_i on the grid) we reflect the diffusion back into the grid if it tries to leave. Using a Markov chain approximation technique we only have 2 choices for the boundary condition. Either we reflect or we give the value function some numerical value at the boundary (absorption). Since we do not know the value function at the boundary we use a reflecting boundary condition. If $u_i = Y_{max}$ then we reflect the diffusion in a simple way. We amend the transition (2.4.13) probabilities to read

$$\begin{aligned}
p\left(\begin{pmatrix} u_i \\ v_j \end{pmatrix}, \begin{pmatrix} u_i - \Delta u \\ v_j \end{pmatrix}\right) &= \left(\frac{b(x, h(x), c(x))^-}{\Delta u} + \frac{\kappa(x)}{(\Delta u)^2}\right)/Q(x) \\
p\left(\begin{pmatrix} u_i \\ v_j \end{pmatrix}, \begin{pmatrix} u_i + \Delta u \\ v_j \end{pmatrix}\right) &= 0 \\
p\left(\begin{pmatrix} u_i \\ v_j \end{pmatrix}, \begin{pmatrix} u_i \\ v_j - \Delta v \end{pmatrix}\right) &= \frac{h(x)^-}{\Delta v}/Q(x) \\
p\left(\begin{pmatrix} u_i \\ v_j \end{pmatrix}, \begin{pmatrix} u_i \\ v_j + \Delta v \end{pmatrix}\right) &= \frac{h(x)^+}{\Delta v}/Q(x) \\
p\left(\begin{pmatrix} u_i \\ v_j \end{pmatrix}, \begin{pmatrix} u_i \\ v_j \end{pmatrix}\right) &= \left(Q(x) - \left(\frac{b(x, h(x), c(x))^-}{\Delta u} + \frac{|h(x)|}{\Delta v^d} + \frac{\kappa(x)}{(\Delta u)^2}\right)\right)/Q(x)
\end{aligned} \tag{2.4.16}$$

2.4.4 Results

There are two main sources of error in the numerical calculation. One is a discretisation error and the other is an error due to having a reflecting upper boundary. Having more resolution can improve the discretisation error but will have no effect on the error induced from the reflecting upper boundary. We can see the latter effect by lifting the boundary and observing the effect on the results.

We are limited to a grid size of about 301X301 points due to the limitation on the RAM of the computer used (512MB). Using this size grid and the parameters chosen for figure (2-14) (see below) the calculation takes about 3 hours to run when we set the criteria to stop calculating further iterations as the maximum difference of the value function between iterations to be 0.001 and to have at least 10 iterations (this is the criteria we use to end all calculations). The initial starting controls are h set to 0 everywhere and consumption set to that of the classical Merton solution.

Firstly we look at a limiting case where we know a closed form solution to the problem to demonstrate that the code is working. Numerically, we cannot take the limiting case as ε goes to zero since the h control gets too large but we can look at the case where $\varepsilon \rightarrow \infty$ by setting ε in the code to larger and larger

(but finite) values. In this case it becomes prohibitively expensive to invest in the share and in the limit the agent consumes optimally from his cash holding. This gives a value function we can solve for in closed form, we set $\mu = r$ in the classical Merton problem with the wealth equal to the cash holding to get

$$\begin{aligned} V_M(Y) &= \left(\frac{\rho + (R-1)r}{R}\right)^{-R} U(Y) \\ C(Y) &= \left(\frac{\rho + (R-1)r}{R}\right) Y \end{aligned} \tag{2.4.17}$$

As we increase ε we expect the H dependence on the value function to get less and less. This means for very large ε we can treat the problem as effectively 1 dimensional and not have much resolution in the H direction and lots more in the Y direction. This enables us to estimate the error due to discretisation. Using smaller values of ε the problem is truly 2 dimensional and we need a reasonable resolution in the H direction. The discretisation error is likely to be higher in that case.

To produce figure (2-6) the parameter values used were $r = 0.04$, $\mu = 0.07$, $\sigma = 0.5$, $\rho = 0.5$ and $R = 2/3$. This gives a Merton proportion, $\pi_* = 0.18$. Y values range from $Y_{\min} = 0$ to $Y_{\max} = 100$ with 300 steps and the H values range from $H_{\min} = 0$ to $H_{\max} = 0.18/(1 - 0.18) * 20$ with 20 steps. Figure (2-6) shows the effect on the value function of increasing ε from 0.1 to 10000 at a fixed Y value of 50 and at different H values. The value function is an increasing function of H but we can see the dependence on H getting less and less as we increase ε . For $\varepsilon = 10000$ the value function changes by 0.0018 from $H = 0$ to the first H step and thereafter is virtually constant the difference being the order of 10^{-12} . This can be explained since ε is so large the incremental benefit of increasing the initial H to values after the first H step is so far in the future that with a $\rho = 0.5$ after discounting it is negligible. The decrease in the value function from increasing ε from 1000 to 10000 with a non zero H is approximately constant at 0.0025.

Figure (2-7) shows the effect on the value function at $Y = 50$ and $H = 2.195$ with $\varepsilon = 10,000$ (other parameters as for figure (2-6)) of increasing the resolution in the Y direction from 300 steps to 1800. It is increasing with the rate of increase decreasing. We are interested in finding the limiting value. Kushner and Dupuis (1992) state that using a Richardson extrapolation with order of convergence 1

generally produces good results but there is no theory to support this practise in general. We estimate the order of convergence, k , by the average obtained from using 2 sets of grid triples with 1200, 2400 and 4800 steps, and 900, 1800 and 3600 steps¹, only trusting the values when they are similar. Column 3 of Table (2.1) shows the convergence order obtained for various values of Y with $H = 2.195$. The first result corresponds to (900, 1800 and 3600) steps and the second to (1200, 2400 and 4800) steps. Column 4 gives the extrapolated value using grid refinements with 4800 steps and 2400 steps². We compare the extrapolated values with the value obtained using 300 steps (column 5) by giving the percentage difference in column 6. We also show the result obtained using (2.4.17) in column 2. We see that the extrapolated values are very close to the values obtained from (2.4.17) with the percentage error using a coarse grid of 300 steps increasing as we decrease the Y values. Table (2.2) shows the results at $H = 0$ and tells a similar story. The percentage errors when we use a grid of 301*301 points with smaller values for ε are likely to be greater than those obtained here since the problem is then truly 2-dimensional.

Figure (2-8) compares the consumption obtained from the numerical study with 4800 steps in the Y direction with that from (2.4.17) ($\varepsilon = 10000$ at $H = 0$, other parameters the same as for figure (2-6)). We see we have good agreement for about half the grid until the upper boundary condition causes consumption to increase rapidly to its maximum allowed value in the code. The percentage difference at $Y = 30$ is 0.14% at $Y = 50$ it is 1.09% and at $Y = 70$ it is 10.26%.

¹given 3 values on successive grid refinements with half the spacing, f_1, f_2 and f_3 , you can calculate the order of convergence as $k = -\log(\frac{f_3-f_2}{f_2-f_1})/\log(2)$

²The extrapolated value from 2 values f_1 and f_2 from successive grid refinements (the second with half the spacing of the first) with an order of convergence k is $f_2 + \frac{f_2-f_1}{2^k-1}$

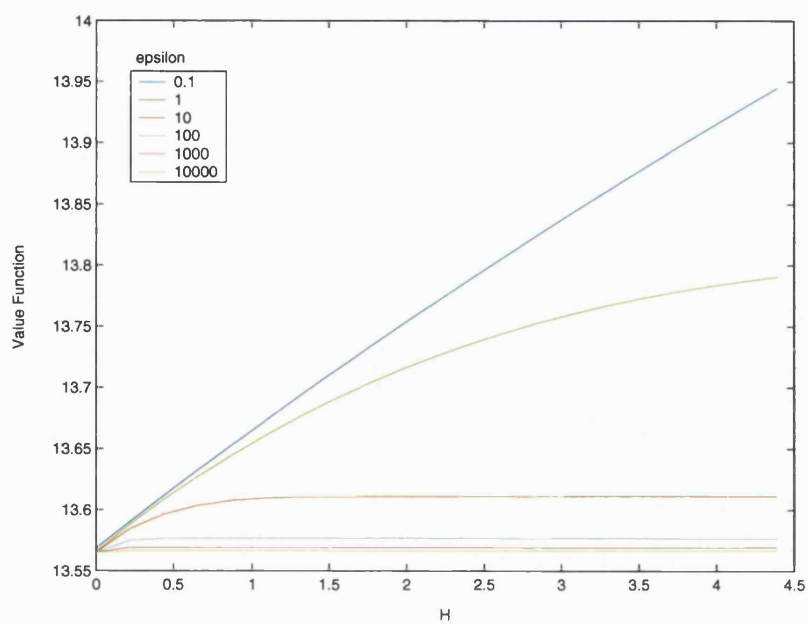


Figure 2-6: Increasing ε at $Y = 50$

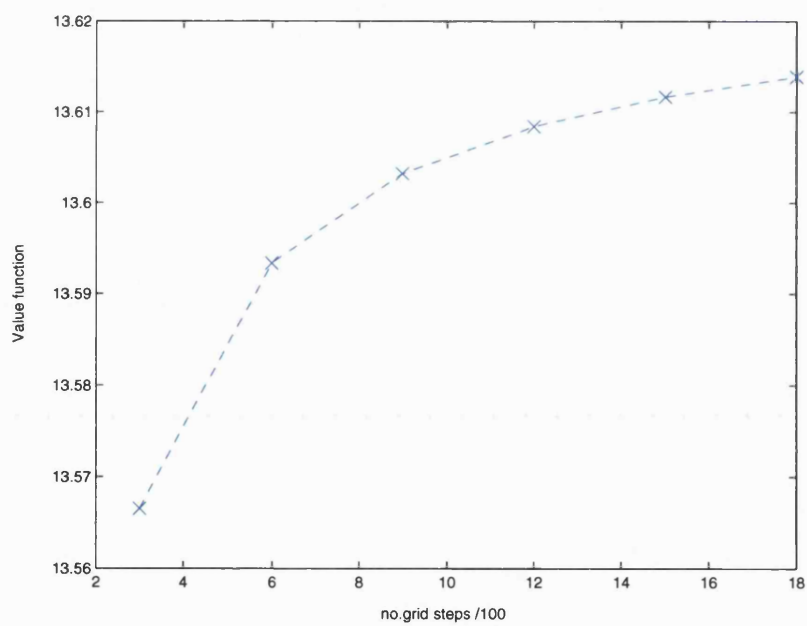


Figure 2-7: Increasing grid steps at $Y = 50$, $H = 2.195$

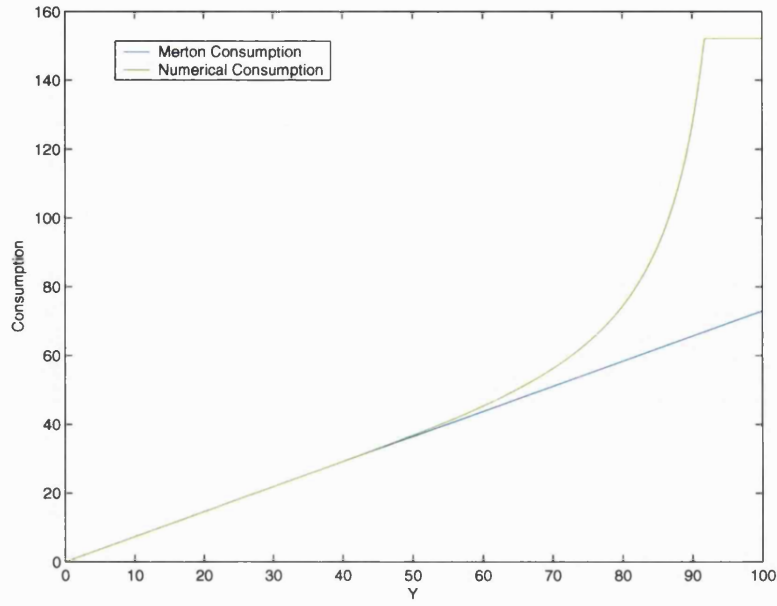


Figure 2-8: Consumption at $H = 0$, 4800 grid steps in Y direction

Y	Merton Value	Convergence Order	Limit (R extrapol)	300 steps	error%
10	7.9721	0.848, 0.859	7.973	7.834	1.74
20	10.0442	0.853, 0.864	10.045	9.948	0.97
30	11.4978	0.856, 0.867	11.498	11.420	0.68
40	12.6549	0.859, 0.864	12.654	12.587	0.53
50	13.6321	0.862, 0.845	13.627	13.567	0.44

Table 2.1: Table showing discretisation error at $H=2.195$

Y	Merton Value	Convergence Order	Limit (R extrapol)	300 steps	error%
10	7.9721	0.840, 0.850	7.972	7.829	1.79
20	10.0442	0.845, 0.856	10.045	9.945	1.00
30	11.4978	0.849, 0.859	11.498	11.417	0.70
40	12.6549	0.852, 0.857	12.654	12.584	0.55
50	13.6321	0.855, 0.837	13.627	13.565	0.45

Table 2.2: Table showing discretisation error at $H=0$

Now we use a grid size of 301*301 points with $\varepsilon = 0.1$ and $\varepsilon = 0.01$, Y ranges from $Y_{\min} = 0$ to $Y_{\max} = 100$ and H ranges from $H_{\min} = 0$ to $H_{\max} = 0.18/(1 - 0.18) * 100$ (other parameters the same as to produce figure (2-6)) . The upper boundary condition will affect the result so we do an additional run with $\varepsilon = 0.1$ and an upper boundary of $Y_{\max} = 100 * 2/3$ with 200 steps (other parameters the same). Figure (2-9) shows the H - Y space is divided into 2 regions a buy and a sell region indicated by 'B' and 'S' and the partition line where $h = 0$. The Merton line shows the position in the classical liquid case. For $\varepsilon = 0.1$, for a given holding of the stock you would like to hold more cash than in the classical case with that level being less if $\varepsilon = 0.01$ but still greater than the classical case except where the upper boundary affects the results. Looking at what happens when we lift the upper boundary from $2/3 * 100$ to 100 we see that the effect of the upper boundary is to cause the partition line to flatten. Figure (2-10) shows the value function at a value of $H = 10.9756$. The value function is higher for $\varepsilon = 0.01$ than it is for $\varepsilon = 0.1$ as you would expect. Looking at the effect of lifting the value of Y_{\max} we see that there is a departure near the upper boundary (red line compared to the blue line) but it is not as marked as the effect on consumption in figure (2-11). This is not surprising since the consumption depends on the derivative of the value function with respect to Y . The percentage difference in consumption after lifting the boundary in figure (2-11) at $Y = 50$ is 12.57% and at $Y = 30$ it is 0.39% where-as for the value function the percentages are 0.31% and 0.008% respectively. Figure (2-12) shows the effect of varying ρ (with 100 steps in the H direction, $H_{\min} = 0$, $H_{\max} = \frac{0.18}{1-0.18} * 100$, 300 steps in the Y direction with $Y_{\min} = 0$ and $Y_{\max} = 100$, $\varepsilon = 0.1$ and the other parameters the same as for figure (2-9)) on the value function. Figure (2-13) shows the effect of increasing μ from 0.07 to 0.2 ($\mu = 0.2$ gives $\pi_* = 0.96$) with $\rho = 0.3$ (other parameters as for figure (2-12)). Finally, figures (2-14), (2-15) and (2-16) show the surface of value function, h and consumption in the H - Y plane for the numerical run with 301*301 points and $\varepsilon = 0.1$ (other parameters as for figure (2-9)) displaying only the first 101 points in the Y and H direction.

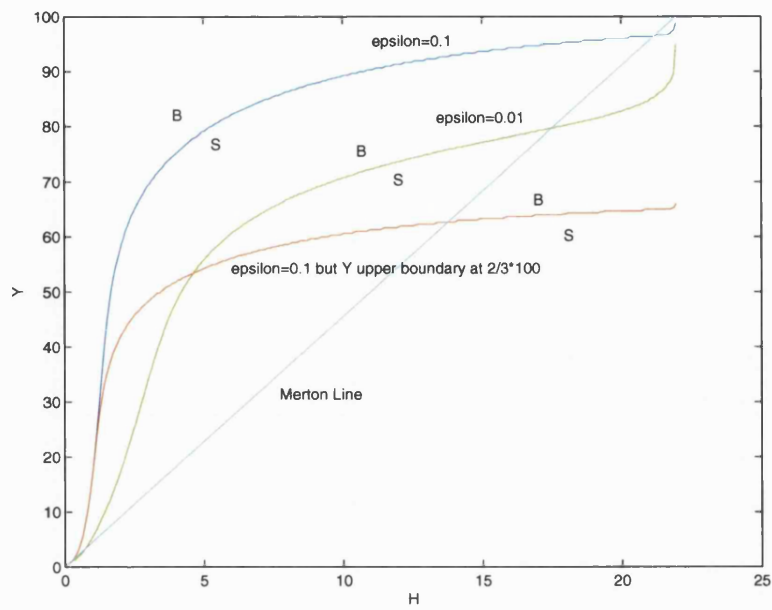


Figure 2-9: Buy Sell region with Merton Line, 301X301 points

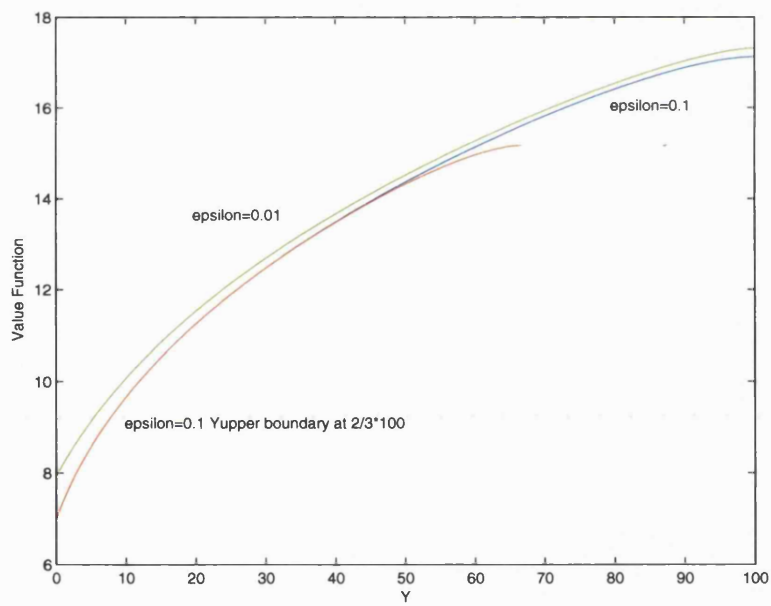


Figure 2-10: Value function comparison at $H=10.9756$

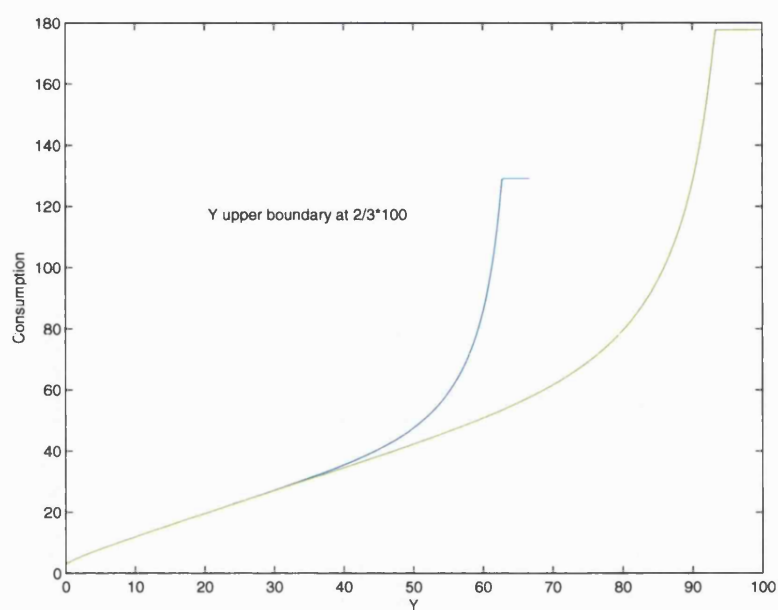


Figure 2-11: boundary effect on consumption $\varepsilon = 0.1$ $H=10.9786$

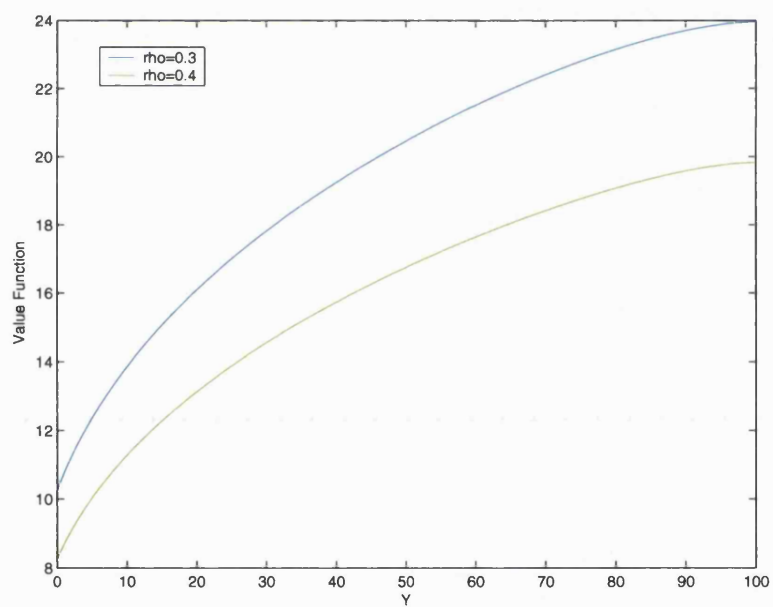


Figure 2-12: effect of varying ρ at $H=10.9756$

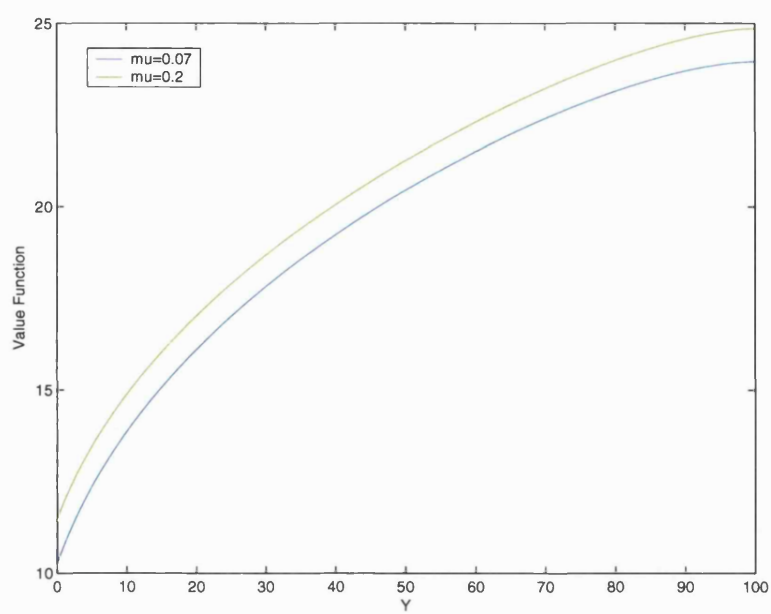


Figure 2-13: effect of varying π_* at $H=10.9756$

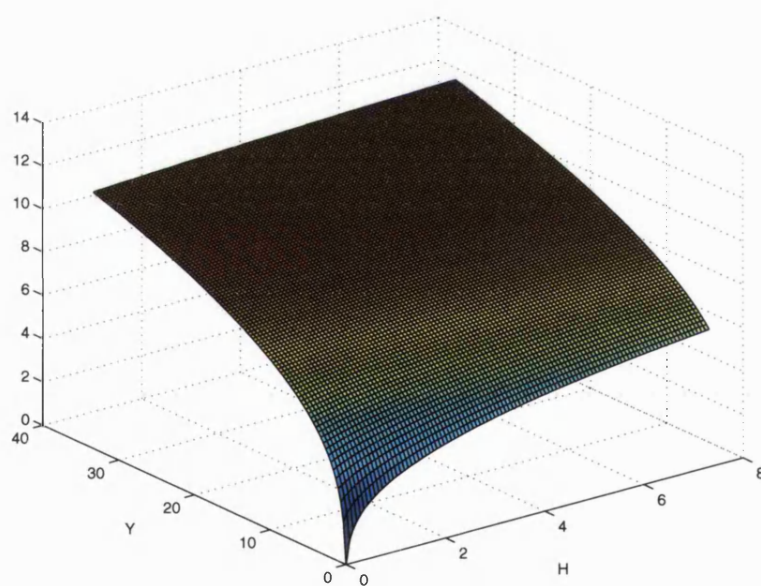


Figure 2-14: Value function, 101X101 points, $\varepsilon = 0.1$

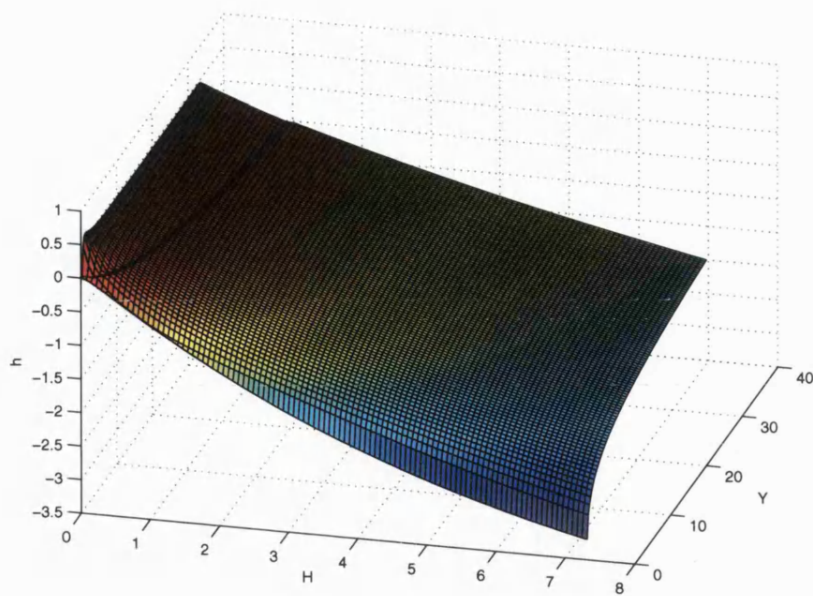


Figure 2-15: h , 101X101 points, $\varepsilon = 0.1$

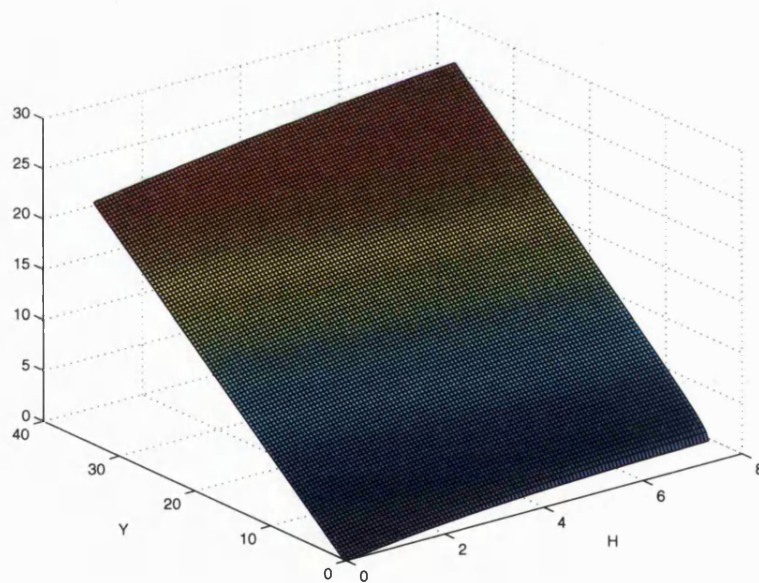


Figure 2-16: Consumption, 101X101 points, $\varepsilon = 0.1$

2.5 Further Scaling

Refering back to (2.3.2) and (2.3.3) for a power utility there is a further scaling present. Making the dependence on ε explicit.

$$v(\lambda z, \lambda H; \frac{\varepsilon}{\lambda}) = \lambda^{1-R} v(z, H; \varepsilon) \quad (2.5.1)$$

To see this note that an admissible strategy (c, h) starting at z_0, H_0 and ε is admissible if and only if the strategy $(\lambda c, \lambda h)$ starting at $\lambda z_0, \lambda H_0$ and $\frac{\varepsilon}{\lambda}$ is admissible where $\lambda > 0$.

Let $\lambda = \frac{1}{z}$ then (2.5.1) becomes

$$v(z, H; \varepsilon) = z^{1-R} F(U, x) \quad (2.5.2)$$

where $F(U, x) = v(1, U, x)$, $U = H/z$ and $x = z\varepsilon$. It is interesting that there is no dependence on ε alone it is the variable $z\varepsilon$ which is important.

For a log utility, the scaling gives in this case

$$F(\lambda z, \lambda H; \frac{\varepsilon}{\lambda}) = F(z, H; \varepsilon) + \frac{\log(\lambda)}{\rho} \quad (2.5.3)$$

Substituting $\lambda = \frac{1}{z}$ gives

$$F(z, H; \varepsilon) = F(1, \frac{H}{z}; z\varepsilon) + \frac{\log(z)}{\rho} \quad (2.5.4)$$

2.6 A Simplified Control Problem

We cannot solve the Merton problem in an illiquid market in closed form but after making some simplifying assumptions and using an heuristic argument we arrive at a control problem we can solve. If we have the following dynamics of the wealth of an investor

$$dw_t = (rw_t - c_t - \eta_t)dt + \theta_t(\sigma dW_t + (\mu - r)dt) \quad (2.6.1)$$

where η_t is a “small liquidity cost”, and the usual objective

$$V(w) = \sup \mathbb{E} \left[\int_0^\infty e^{-\rho s} U(c_s) ds | w_0 = w \right] \quad (2.6.2)$$

Using the principle of optimality the process

$$y_t^{\theta, c} = e^{-\rho t} V(w_t) + \int_0^t e^{-\rho s} U(c_s) ds \quad (2.6.3)$$

is a martingale under optimal control (θ^*, c^*) and a supermartingale under any other control (θ, c) . The loss of objective i.e. $\mathbb{E}[y_\infty^{(\theta^*, c^*)} - y_\infty^{(\theta, c)}]$ is

$$\mathbb{E} \left[\int_0^\infty -e^{-\rho s} \left(U(c_s) - \rho V + \frac{1}{2} \theta_s^2 \sigma^2 V'' + (r w_s - c_s - \eta_s + \theta_s (\mu - r)) V' \right) ds \right] \quad (2.6.4)$$

If we make the approximation $V(w) \approx V_M(w)$ where $V_M(w)$ is the value function in the classical infinite horizon Merton investment-consumption problem and minimise the integrand of (2.6.4) over consumption giving $c_s = (V_M')^{-\frac{1}{R}}$ we have

$$\begin{aligned} & \inf_{c_s} - \left\{ U(c_s) - \rho V + \frac{1}{2} \theta_s^2 \sigma^2 V'' + (r w_s - c_s - \eta_s + \theta_s (\mu - r)) V' \right\} \\ & \approx \inf_{c_s} - \left\{ U(c_s) - \rho V_M + \frac{1}{2} \theta_s^2 \sigma^2 V_M'' + (r w_s - c_s - \eta_s + \theta_s (\mu - r)) V_M' \right\} \\ & = - \left\{ \frac{R}{1-R} (\gamma w_s)^{1-R} - \rho \gamma^{-R} \frac{w_s^{1-R}}{1-R} - \frac{\eta_s}{w} \gamma^{-R} w_s^{1-R} - \frac{R}{2} \left(\frac{\theta_s}{w_s} \right)^2 \sigma^2 \gamma^{-R} w_s^{1-R} \right. \\ & \quad \left. + \left(r + \frac{\theta_s}{w_s} (\mu - r) \right) \gamma^{-R} w_s^{1-R} \right\} \\ & = - \gamma^{-R} w_s^{1-R} \left(\frac{\gamma R - \rho}{1-R} - \frac{\eta_s}{w_s} - \frac{1}{2} \sigma^2 R \left(\frac{\theta}{w_s} \right)^2 + \frac{\theta_s}{w_s} (\mu - r) + r \right) \\ & = \gamma^{-R} w_s^{1-R} \left(\frac{\eta_s}{w_s} + \frac{1}{2} \sigma^2 R (\pi_* - \frac{\theta_s}{w_s})^2 \right) \end{aligned} \quad (2.6.5)$$

We let w_t and z_t follow the process in the classical Merton problem.

$$\begin{aligned} w_t &= w_0 \exp(\sigma \pi W_t + \kappa t) \\ z_t &= z_0 \exp(-\sigma(1 - \pi_*) W_t + \nu t) \end{aligned} \quad (2.6.6)$$

where $\kappa = r + \pi_*(\mu - r) - \gamma - \frac{1}{2}\sigma^2\pi_*^2$ and $\nu = \kappa - \mu + \frac{1}{2}\sigma^2$. We use the liquidity cost $\eta_t = \epsilon h_t^2 \frac{S_t}{z_t} z_0$ and $H_t = \theta_t S_t$ in (2.6.5). The control problem we will solve is

$$V(z, w, H) = \inf_h \mathbb{E}_{\mathbb{P}} \left[\int_0^\infty \gamma^{-R} e^{-\rho t} w_t^{1-R} \left(\epsilon \frac{h_t^2}{z_t^2} z_0 + \frac{1}{2} \sigma^2 R \left(\pi_* - \frac{H_t}{z_t} \right)^2 \right) dt \middle| z_0 = z, w_0 = w, H_0 = H \right] \quad (2.6.7)$$

Now, this is a problem we can solve in closed form. Firstly we change measure to get rid of the w_t^{1-R} on the right hand side of (2.6.7). Define a new measure \mathbb{Q} by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \exp(-\pi_*(R-1)\sigma W_t - \frac{1}{2}\pi_*^2(R-1)^2\sigma^2 t) \quad (2.6.8)$$

and

$$z_t = z_0 \exp(-\sigma(1-\pi_*)W'_t + (\nu - \sigma^2\pi_*(1-\pi_*)(1-R))t) \quad (2.6.9)$$

where W'_t is a \mathbb{Q} brownian motion. (2.6.7) becomes

$$\begin{aligned} V(z, w, H) &= w^{1-R} \inf_h \mathbb{E}_{\mathbb{Q}} \left[\int_0^\infty \gamma^{-R} e^{-\rho' t} \left(\epsilon \frac{h_t^2}{z_t^2} z_0 + \frac{1}{2} \sigma^2 R \left(\pi_* - \frac{H_t}{z_t} \right)^2 \right) dt \middle| z_0 = z, H_0 = H \right] \\ &= w^{1-R} F(z, H) \end{aligned} \quad (2.6.10)$$

where $\rho' = \rho - \kappa(1-R) - \frac{1}{2}\pi_*^2(R-1)^2\sigma^2$.

There is a further scaling property present here. For $\lambda > 0$

$$F(\lambda z, \lambda H) = F(z, H) \quad (2.6.11)$$

Taking $\lambda = 1/z$ gives

$$G(U) = F(1, \frac{H}{z}) = F(z, H) \quad (2.6.12)$$

where $U = \frac{H}{z}$. Let $\alpha_t = \frac{h_t}{z_t}$ and applying Ito's formula to U_t gives (note: $\rho' = \sigma^2\pi_*(1-\pi_*)(1-R) + \frac{1}{2}\sigma^2(1-\pi_*)^2 - \nu$)

$$dU_t = U_t(\sigma(1-\pi_*)dW'_t + \rho' dt) + \alpha_t dt \quad (2.6.13)$$

with the objective

$$G(U) = \inf_\alpha \mathbb{E}_{\mathbb{Q}} \left[\int_0^\infty \gamma^{-R} e^{-\rho' t} \left(\epsilon \alpha_t^2 + \frac{1}{2} \sigma^2 R (\pi_* - U_t)^2 \right) dt \middle| U_0 = U \right] \quad (2.6.14)$$

Assuming $\rho' > 0$ this is a linear quadratic optimal control problem with an infinite horizon. Yong and Zhou (1999) study linear quadratic control problems on a finite horizon in considerable detail. They consider the more general case where the control can appear in the coefficient of dW_t in (2.6.13). They also allow a terminal cost at T , where T is the horizon of the problem. They show that by conjecturing a quadratic solution in the space variable and substituting into the HJB equation, you arrive at a system of equations, which when solved with the appropriate boundary condition from the control problem at time T , is the value function. We do the corresponding finite horizon case of (2.6.14) in the appendix. Reassuringly the limit of the solution in the appendix as $T \rightarrow \infty$ is the solution we arrive at for the infinite horizon case below in (2.6.19). In the infinite horizon case we do not have any terminal boundary condition. Instead a sufficient condition for optimality of a solution to the HJB, $G(U)$, and the corresponding control, α^* , is the requirement that $\lim_{t \rightarrow \infty} \mathbb{E}[e^{-\rho' t} G(U_t)] = 0$ for any admissible control (see for example Chang (2004) or Fleming and Soner (2005)). We verify the solution (2.6.19) by comparing with numerical results for a range of parameter values.

The Hamilton-Jacobi-Bellman equation for this problem is.

$$\inf_{\alpha} \left\{ \gamma^{-R} (\epsilon \alpha^2 + \frac{1}{2} \sigma^2 R (\pi_* - U)^2) + (\rho' U + \alpha) G_U + \frac{1}{2} \sigma^2 (1 - \pi_*)^2 U^2 G_{UU} - \rho' G \right\} = 0 \quad (2.6.15)$$

which gives

$$\alpha^* = -\frac{G_U}{2\epsilon\gamma^{-R}} \quad (2.6.16)$$

Substituting (2.6.16) into (2.6.15) yields

$$2\epsilon\gamma^{-2R}\sigma^2R(\pi_* - U)^2 + 4\epsilon\gamma^{-R}\rho'UG_U + 2\epsilon\gamma^{-R}\sigma^2(1 - \pi_*)^2U^2G_{UU} - 4\epsilon\gamma^{-R}\rho'G - G_U^2 = 0 \quad (2.6.17)$$

If we set the coefficients

$$\begin{aligned}
a &= 2\varepsilon\gamma^{-2R}\sigma^2R \\
b &= 4\varepsilon\gamma^{-R}\rho' \\
c &= 2\varepsilon\gamma^{-R}\sigma^2(1 - \pi_*)^2
\end{aligned}
\tag{2.6.18}$$

The solution is

$$G(U) = AU^2 + BU + C \tag{2.6.19}$$

where

$$A = \frac{b + 2c + \sqrt{16a + (b + 2c)^2}}{8} \tag{2.6.20}$$

$$B = \frac{1}{4}(b + 2c - \sqrt{16a + (b + 2c)^2})\pi_* \tag{2.6.21}$$

$$C = -\frac{(b + 2c)(b + 2c - \sqrt{16a + (b + 2c)^2})}{8b}(\pi_*)^2 \tag{2.6.22}$$

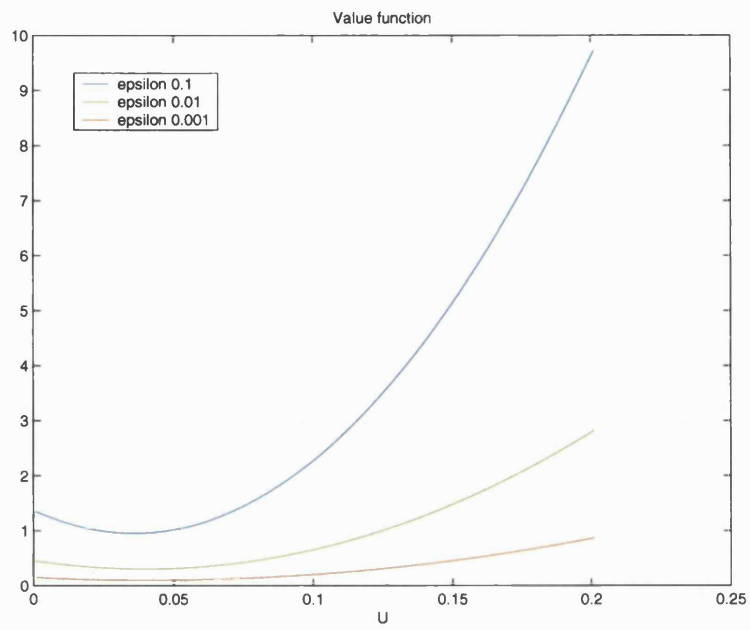


Figure 2-17: Effect of varying epsilon

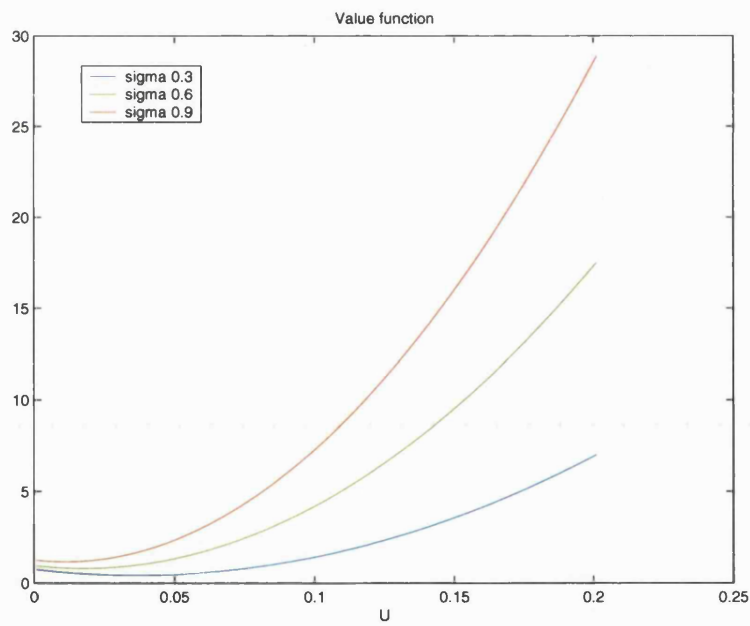


Figure 2-18: Effect of varying sigma

For figure (2-17), $R = 3$, $\sigma = 0.4$, $\mu = 0.09$, $r = 0.07$ and $\rho = 0.1$. The minimum of the value function isn't given by the Merton proportion. The minimum of the value function for $\epsilon = 0.1$, $\epsilon = 0.01$ and $\epsilon = 0.001$ is 0.0360, 0.0398 and 0.0411 respectively where-as the Merton proportion for this problem is 0.0417. For figure (2-18) we vary σ but keep the market price of risk, $\eta = \frac{\mu-r}{\sigma}$ and γ fixed. The parameters are $R = 3$, $\epsilon = 0.1$, $\eta = 0.0375$, $r = 0.07$, $\rho = 0.1$

2.6.1 Numerical Solution

We use a Markov chain approximation method to solve the problem (2.6.14). This is much simpler than the numerical problem of section (2.4) since it is 1-dimensional. We define the state space $S = \{u_i : i \in \{1, 2, \dots, n\}\}$, $S_0 = \{u_i : i \in \{2, \dots, n-1\}\}$ and $\partial S = S \setminus S_0$. In a similar way to section (2.4) we derive transition probabilities for the chain to be locally consistent with (2.6.13). For $u_i \in S_0$ the transition probabilities we use are

$$\begin{aligned} p(u_i, u_i + \Delta u) &= \left(\frac{f(u_i)}{(\Delta u)^2} + \frac{g(u_i, \alpha)^+}{\Delta u} \right) / Q(u_i) \\ p(u_i, u_i - \Delta u) &= \left(\frac{f(u_i)}{(\Delta u)^2} + \frac{g(u_i, \alpha)^-}{\Delta u} \right) / Q(u_i) \\ p(u_i, u_i) &= 1 - \left(\frac{2f(u_i)}{(\Delta u)^2} + \frac{|g(u_i, \alpha)|}{\Delta u} \right) / Q(u_i) \end{aligned} \quad (2.6.23)$$

where $f(u) = \frac{1}{2}\sigma^2(1 - \pi_*)^2u^2$ and $g(u, \alpha) = \rho'u + \alpha$. $Q(u) = \frac{2f(u)}{(\Delta u)^2} + \frac{g_{max}(u)}{\Delta u}$ where $g_{max}(u) = \rho'u + \alpha_{max}$ with $-\alpha_{max} < \alpha < \alpha_{max}$ in the code.

If we are given $G_k^d(\cdot)$, the value function after the k^{th} policy improvement, the policy improvement algorithm (for $u_i \in S_0$) for the $(k+1)^{th}$ iteration of the control, α_{k+1} , is as follows.

Assume $g(u_i, \alpha_{k+1}) \geq 0$, $\alpha^* = \frac{-e^{-\frac{\rho'}{Q(u_i)}} D^+ G_k^d(u_i)}{2\gamma - R\epsilon}$

If $g(u_i, \alpha^*) \geq 0$ then

$$\alpha_1 = \alpha^*$$

else

$$\alpha_1 = -\rho' u_i.$$

$$\text{Assume } g(u_i, \alpha_{k+1}) \leq 0, \alpha^* = \frac{-e^{-\frac{\rho'}{Q(u_i)}} D^- G_k^d(u_i)}{2\gamma^{-R}\varepsilon}$$

If $g(u_i, \alpha^*) \leq 0$ then

$$\alpha_2 = \alpha^*$$

else

$$\alpha_2 = -\rho' u_i.$$

$$\alpha_{k+1} = \operatorname{argmax}_{\alpha \in \{\alpha_1, \alpha_2\}} \left\{ \gamma^{-R} (\varepsilon \alpha^2 + \frac{1}{2} \sigma^2 R (\pi_* - u_i)^2) / Q(u_i) + \sum_{v \in S} e^{\frac{-\rho'}{Q(u_i)}} p(u_i, v) G_k^d(v) \right\}$$

For $u_i \in \partial S$ we assume $g(u_i, \alpha^{k+1}) \geq 0$ and we show results assuming a reflecting upper boundary, figure (2-19), and results using an upper boundary condition from the closed form solution (2.6.19) i.e. we set the value function at $U = 1$ obtained from evaluating (2.6.19) at $U = 1$, figure (2-20). The values for the parameters are $R = 3$, $r = 0.07$, $\varepsilon = 0.1$, $\rho = 0.1$, $\mu = 0.09$, $\sigma = 0.4$ and $0 \leq U \leq 1$ with 201 points in the state space. The stopping criterion is when the maximum difference between values on all grid points of successive iterations is < 0.001 . When using a reflecting upper boundary condition figure (2-19) shows we have good agreement except close to the boundary where the value function is less than the theoretical one. This is not surprising since the markov chain is artificially constrained to stay in the region $0 \leq U \leq 1$. When using an upper boundary condition taken from the closed form solution figure (2-20) shows we get good agreement throughout $0 \leq U \leq 1$.

Table (2.3) shows the percentage error between the numerical solution using the reflecting boundary condition and 1000 steps and the closed form solution (2.6.19) for $U = 0.2$, $U = 0.4$ and $U = 0.6$. The first column shows the parameter changed relative to the base case to produce figure (2-19). In all cases the error is less than a percent. Table (2.4) shows the percentage error but this time using 2000 steps. In all cases the error is less than 0.5 percent.

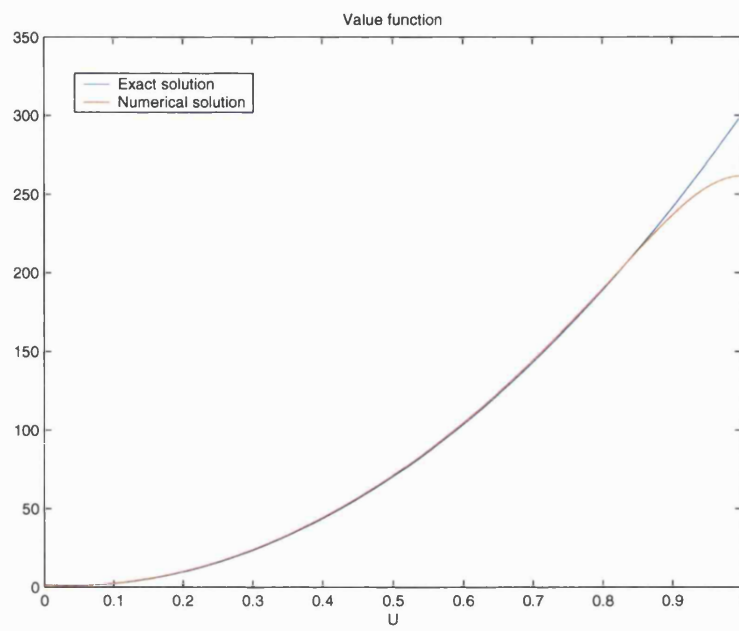


Figure 2-19: Comparison of (2.6.19) and numerical solution (reflecting boundary condition)

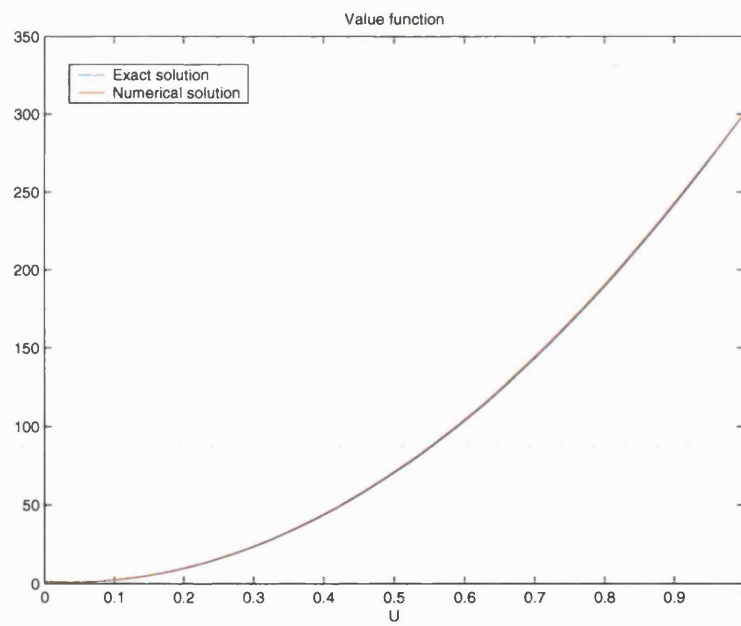


Figure 2-20: Comparison of (2.6.19) and numerical solution (upper boundary condition taken from (2.6.19))

parameter	U=0.2 (%)	U=0.4 (%)	U=0.6 (%)
no change	0.875	0.330	0.197
$\varepsilon = 0.2$	0.771	0.305	0.156
$\varepsilon = 0.3$	0.715	0.289	0.045
$\rho = 0.2$	0.755	0.299	0.182
$\rho = 0.3$	0.681	0.279	0.172
$R = 4$	0.813	0.317	0.193
$R = 5$	0.768	0.308	0.189
$\mu = 0.07$	0.477	0.238	0.157
$\mu = 0.08$	0.663	0.281	0.176

Table 2.3: % error of numerical routine using 1000 steps

parameter	U=0.2 (%)	U=0.4 (%)	U=0.6 (%)
no change	0.438	0.165	0.098
$\varepsilon = 0.2$	0.386	0.152	0.064
$\varepsilon = 0.3$	0.358	0.144	0.042
$\rho = 0.2$	0.378	0.149	0.090
$\rho = 0.3$	0.340	0.140	0.085
$R = 4$	0.407	0.159	0.096
$R = 5$	0.384	0.154	0.095
$\mu = 0.07$	0.239	0.119	0.078
$\mu = 0.08$	0.332	0.141	0.088

Table 2.4: % error of numerical routine using 2000 steps

2.7 Asymptotics

Here we find an asymptotic expansion for (2.6.19) for small ε and compare it with the expansion obtained from the pde (2.6.17) by posing a power expansion in powers of $\sqrt{\varepsilon}$. Expanding (2.6.19) in powers of ε to $O(\varepsilon)$ gives

$$\begin{aligned} & \left(\frac{(U - \pi_*)^2 \sqrt{R \sigma^2}}{\sqrt{2} \gamma^R} + \frac{(1 - \pi_*)^2 \pi_*^2 \sigma^2 \sqrt{R \sigma^2}}{\sqrt{2} \gamma^R \rho'} \right) \sqrt{\varepsilon} \\ & + \left(\frac{U (U + 2 \pi_*) (\rho' + (-1 + \pi_*)^2 \sigma^2)}{2 \gamma^R} - \frac{(\pi_*^2 (\rho' + (-1 + \pi_*)^2 \sigma^2)^2)}{2 \gamma^R \rho'} \right) \varepsilon + O(\varepsilon^{3/2}) \end{aligned} \quad (2.7.1)$$

We pose the expansion

$$G(U) = f(U) \sqrt{\varepsilon} + g(U) \varepsilon + h(U) \varepsilon^{3/2} + O(\varepsilon^2) \quad (2.7.2)$$

and substitute into (2.6.17). We then set the coefficients of each power of ε to zero to find the unknown functions $f(U)$ and $g(U)$. The coefficient of ε gives the ode

$$\left(\frac{2 R (-U + \pi_*)^2 \sigma^2}{\gamma^{2R}} - f'(U)^2 \right) = 0 \quad (2.7.3)$$

Solving

$$f(U) = \frac{(U - \pi_*)^2 \sqrt{R \sigma^2}}{\sqrt{2} \gamma^R} + \kappa \quad (2.7.4)$$

(We take the positive root since we know the solution to the control problem is non-negative) Substituting (2.7.4) into (2.7.2) and back into (2.6.17) gives the following for the coefficient of $\varepsilon^{3/2}$

$$\frac{-2 \left(2 \gamma^R \kappa \rho' + \sqrt{2} \sqrt{R \sigma^2} (\pi_*^2 \rho' - U^2 (\rho' + (-1 + \pi_*)^2 \sigma^2)) + \sqrt{2} \gamma^R (U - \pi_*) \sqrt{R \sigma^2} g'(U) \right)}{\gamma^{2R}} \quad (2.7.5)$$

Setting $U = \pi_*$ and setting the coefficient to 0 gives the following value for κ

$$\kappa = \frac{(1 - \pi_*)^2 \pi_*^2 \sigma^2 \sqrt{R \sigma^2}}{\sqrt{2} \gamma^R \rho'} \quad (2.7.6)$$

Using this value of κ in (2.7.5) solving the differential equation gives

$$g(U) = \frac{U (U + 2\pi_*) (\rho' + (-1 + \pi_*)^2 \sigma^2)}{2\gamma^R} + \beta \quad (2.7.7)$$

Substituting back into (2.7.2) and back into (2.6.17) the coefficient of ε^2 is

$$- \left(\frac{\left(4\beta\gamma^R\rho' - (U^2 - 2U\pi_* - \pi_*^2) (\rho + (-1 + \pi_*)^2 \sigma^2)^2 + 2\sqrt{2}\gamma^R (U - \pi_*) \sqrt{R\sigma^2} h'(U) \right)}{\gamma^{2R}} \right) \quad (2.7.8)$$

Putting $U = \pi_*$ and setting (2.7.8) to zero gives a value for β

$$\beta = \frac{- \left(\pi_*^2 (\rho' + (-1 + \pi_*)^2 \sigma^2)^2 \right)}{2\gamma^R \rho'} \quad (2.7.9)$$

Putting all this together we have

$$G(U) = \left(\frac{(U - \pi_*)^2 \sqrt{R\sigma^2}}{\sqrt{2}\gamma^R} + \frac{(1 - \pi_*)^2 \pi_*^2 \sigma^2 \sqrt{R\sigma^2}}{\sqrt{2}\gamma^R \rho'} \right) \sqrt{\varepsilon} + \left(\frac{U (U + 2\pi_*) (\rho' + (-1 + \pi_*)^2 \sigma^2)}{2\gamma^R} - \frac{(\pi_*^2 (\rho' + (-1 + \pi_*)^2 \sigma^2)^2)}{2\gamma^R \rho'} \right) \varepsilon + O(\varepsilon^{3/2}) \quad (2.7.10)$$

which agrees with the expansion (2.7.1)

2.8 Conclusions

In the first chapter we highlighted various attempts at modelling liquidity and observed that the transaction cost element of liquidity hasn't been modelled successfully. Having a large bid-offer spread in the proportional transaction cost model can be used as a proxy for an illiquid market but it doesn't capture the price variation for different amounts purchased. We believe by assuming the share holding process is a finite variation one and by putting a cost on the derivative of that process with respect to time we have found a good way to model transaction

costs in an illiquid market. In this chapter we have looked at a classic problem from the world of finance, the Merton problem of investment and consumption with an infinite horizon. The classical problem is 1 dimensional. Even in the presence of proportional transaction costs there is a scaling property which reduces it down to 1 dimension but with liquidity costs we are stuck with a truly 2 dimensional problem. We use a Markov chain approximation method with policy improvement to get numerical results. We show that by taking ε very large the value function from the numerical routine converges to the case of optimally consuming from the cash holding. For more realistic values of ε using a pc with a 2.67 Ghz processor and 512 Mb RAM it is not really possible to get very accurate results concerning the continuous time limit of the problem but the numerics provide useful qualitative insights into the optimal control h_t . In particular we have two regions, a buy region and a sell region and these regions are *not* separated by the Merton line. A further area for research would be to try and get some quantitative results concerning the boundary of the two regions. The final part of this chapter ends with a related control problem which can be solved in closed form. We also solve the problem using a Markov chain approximation method and compare the results.

Chapter 3

Option Pricing in an Illiquid Market

3.1 Utility Indifference Pricing

To price options in the presence of finite liquidity the usual theory breaks down. It is no longer possible to perfectly replicate the option payoff. The situation is similar to pricing with proportional transaction costs so we look to work done with that model for guidance. The technique we use to find an option price is utility maximisation used by Hodges and Neuberger (1989) and by Davis, Panas and Zariphopoulou (1993).

Here we briefly review the pricing methodology using a put option as an example. We consider a complete market with a share price process $(S_t)_{t \geq 0}$ satisfying

$$dS_t = S_t(\sigma dW_t + \mu dt) \quad (3.1.1)$$

for constants σ and μ , where $(W_t)_{t \geq 0}$ is a standard Brownian motion. The wealth equation is

$$dw_t = \pi_t(\sigma dW_t + \mu dt) + (w_t - \pi_t)r dt \quad (3.1.2)$$

where π_t is the wealth in the risky asset. Let $EP(t, S_t)$ be the price of a put option at time t for a share price, S_t . We define the value function

$$V(t, w, S, \delta) = \sup_{\pi \in \mathcal{A}(w)} \mathbb{E} \left[U(w_T^\pi + \delta EP(T, S_T)) | w_t = w, S_t = S \right] \quad (3.1.3)$$

where $\mathcal{A}(w)$ is the set of admissible trading strategies starting with initial wealth w . We can define a utility indifference bid price p^b via

$$V(t, w - p^b, S, \delta) = V(t, w, S, 0) \quad (3.1.4)$$

and similarly we can define a utility indifference sell price p^s via

$$V(t, w + p^s, S, -\delta, t) = V(t, w, S, 0) \quad (3.1.5)$$

We assume $V(t, w, S, \delta)$ is finite for all $w \in \mathbb{R}$ and monotonically increasing in w and δ .

It is sensible for any pricing methodology used in an incomplete market to give Black Scholes prices when applied to a complete market. This is indeed the case for utility indifference pricing. This is shown in Davis et al. (1993) and Henderson and Hobson (2004). The proof is quite straightforward and relies on the assumption if $\pi \in \mathcal{A}(w)$ and $\pi^* \in \mathcal{A}(w^*)$ then $a\pi + b\pi^* \in \mathcal{A}(aw + bw^*)$ for $a, b \in \mathbb{R}$. Suppose there exists a replicating strategy for $EP(T, S_T)$ which costs p^{bs} to set up and has a corresponding trading strategy π^{bs} . Then

$$\begin{aligned} V(t, w, S, -\delta) &= \sup_{\pi \in \mathcal{A}(w)} \mathbb{E}[U(w_T^\pi - \delta EP(S_T)) | w_t = w, S_t = S] \\ &= \sup_{\tilde{\pi} \in \mathcal{A}(w - \delta p^{bs})} \mathbb{E}[U(w_T^{\tilde{\pi}} + \delta w_T^{\pi^{bs}} - \delta EP(T, S_T)) | w_t = w, S_t = S] \\ &= \sup_{\tilde{\pi} \in \mathcal{A}(w - \delta p^{bs})} \mathbb{E}[U(w_T^{\tilde{\pi}} | w_t = w - \delta p^{bs}, S_t = S)] \\ &= V(t, w - \delta p^{bs}, S, 0) \end{aligned} \quad (3.1.6)$$

It follows that the utility indifference selling price for one put option is p^{bs} . Similarly one can show that the utility indifference bid price for one put option is p^{bs} .

In the next section we will look at the problem in the illiquid market from the writer's (seller's) point of view and we use an exponential utility,

$$U(x) = -\exp(-\gamma x) \quad (3.1.7)$$

In the complete market case there is an explicit solution for the value function,

$$V(t, w, S, -\delta),$$

$$V(t, w, S, -\delta) = -\exp(-\gamma e^{r\tau}(w - \delta EP(t, S)) - \frac{(\mu - r)^2}{2\sigma^2}\tau) \quad (3.1.8)$$

where $\tau = T - t$. The optimal trading strategy π_t^* is

$$\pi_t^* = S\left(\frac{(\mu - r)e^{-r\tau}}{\gamma S\sigma^2} + \delta EP_s(t, S)\right) \quad (3.1.9)$$

which consists of a holding from the Merton wealth problem without the option liability plus the hedge if you had written δ put options.

3.2 Utility Indifference Pricing in an Illiquid Market

In an illiquid market the wealth dynamics are modified to

$$dw_t = \pi_t(\sigma dW_t + \mu dt) + (w_t - \pi_t)rdt - \epsilon h_t^2 S_t dt \quad (3.2.1)$$

and

$$dH_t = h_t dt \quad (3.2.2)$$

It is not as straightforward to define an option price in the presence of liquidity costs as it is in the complete market case, since now, it is not just the total initial notional wealth which is important, but also the ratio of initial wealth in the risky asset to the cash bond. Another factor which is important is the definition of delivery in the option contract. If the contract specifies physical delivery of some holding of an underlying or if the contract specifies the cash equivalent, it will make a difference to the option price in an illiquid market. In a contract such as a put or call option there is a natural definition of physical delivery. Let K equal the strike price. If $S > K$ then the writer of a call option delivers the asset and receives an amount of money K or if $S < K$ the writer of a put option buys the asset for an amount of money K . If the contingent claim was, for example, S^2 then there is no natural definition.

These complications to option pricing occur in the proportional transaction

cost model. Davis et al. (1993) assume that the initial holding is all in the cash bond i.e. there is no asset holding and if we consider a holding of y shares and a share price S they define a liquidation value, $c(S, y)$, where any long positions in the risky asset are sold and any short positions are closed i.e.

$$c(S, y) = \begin{cases} (1 + \lambda)yS & y < 0 \\ (1 - \eta)yS & y \geq 0 \end{cases}$$

(λ and η are parameters of the proportional transaction cost model. The investor pays fractions λ and η on purchase and sale of the risky asset respectively). They consider the writer's price of a call option. The final wealth is the cash value plus the liquidation value of any share holding after adjusting for any option payoff. If $S < K$ then the option expires worthless so there is no adjustment. If $S > K$ then the cash level is increased by K and the share holding is decreased by 1 unit.

Unfortunately, we do not have any simple definition of liquidation value in our illiquid market. You cannot sell/close out your final positions instantaneously since that action would result in an infinite liquidity cost. We will assume that at expiry we have perfect liquidity so that your notional wealth at expiry can be transformed to any ratio of risky asset holding and cash bond with the same notional value without incurring any penalty. Also, we will assume that the initial position is one where there is no holding of the risky asset. If we set $\mu = r$ we are then guaranteed to get option prices greater than Black Scholes ones.

3.3 Hamilton Jacobi Bellman Equation

The utility function we use is $U(x) = -\exp(-\gamma x)$. The choice of this utility function is one of convenience. It results in a reduction in the dimensionality of the problem. We define the value function,

$$\begin{aligned} V(t, H, Y, S, \delta) &\equiv \sup \mathbb{E}[U(H_T S_T + Y_T + \delta EP(T, S_T)) | H_t = H, Y_t = Y, S_t = S] \\ &= \exp(-\gamma e^{rT} Y) V(t, H, 0, S, \delta) \\ &= \exp(-\gamma e^{rT} Y) F(t, H, S, \delta) \end{aligned} \tag{3.3.1}$$

where

$$F(t, H, S, \delta) \equiv V(t, H, 0, S, \delta) \quad (3.3.2)$$

We have to be careful with this choice of value function as pointed out in, for example, Henderson (2002). Being short options with unbounded payoffs may results in a utility of minus infinity, for example if we consider the payoff of a short call $\mathbb{E}U(-(S_T - K)^+) = -\infty$. This is why we consider a single put option.

Given (3.2.1), the wealth invested in the cash bond Y_t evolves as

$$dY_t = (Y_t r - S_t h_t - \epsilon h_t^2 S_t) dt \quad (3.3.3)$$

The Hamilton Jacobi Bellman equation for this problem is therefore

$$\sup_h \left\{ \mu S V_S + \frac{1}{2} \sigma^2 S^2 V_{SS} + h V_H - (S h + \epsilon h^2 S - Y r) V_Y + V_t \right\} = 0 \quad (3.3.4)$$

Using (3.3.1) this becomes

$$\sup_h \left\{ \mu S F_S + \frac{1}{2} \sigma^2 S^2 F_{SS} + h F_H + (S h + \epsilon h^2 S) \gamma e^{r\tau} F + F_t \right\} = 0 \quad (3.3.5)$$

with the optimal control

$$h = -\frac{e^{-r\tau} F_H + S \gamma F}{2\epsilon \gamma S F} \quad (3.3.6)$$

Substituting (3.3.6) into (3.3.5) we get

$$\mu S F_S + \frac{1}{2} S^2 \sigma^2 F_{SS} + F_t = e^{r\tau} \frac{(e^{-r\tau} F_H + \gamma S F)^2}{4\epsilon \gamma S F} \quad (3.3.7)$$

3.4 Asymptotic control

It is extremely unlikely that there is a closed form solution to (3.3.7) so we attempt an asymptotic analysis as ϵ goes to zero. For the proportional transaction model Whalley and Wilmott (1997) perform an asymptotic analysis as the transaction costs go to zero. They are, however, dealing with a stochastic control problem with three regions. A sell region, a buy region and a no transaction region. which is $O(\epsilon^{\frac{1}{3}})$.

We expect the value function to tend to the classical value function of a

complete market as ϵ goes to zero so we look for an asymptotic solution about the classical one. We look for an asymptotic solution of the form

$$F(t, H, S, -1) \sim -\exp(-\gamma e^{r\tau}(HS - EP(t, S)) - \frac{(\mu - r)^2}{2\sigma^2}\tau + \sum_n \epsilon^{n\delta} g^n(t, H, S)) \quad (3.4.1)$$

After substituting (3.4.1) into (3.3.7) we see that the *L.H.S* of (3.3.7) is to first order independent of ϵ so we take $\delta = \frac{1}{2}$. If we also use the fact that $EP(t, S)$ satisfies the Black Scholes equation i.e. $\frac{1}{2}\sigma^2 S^2 EP_{ss} + rSEP_s - rEP + EP_t = 0$ we get

$$\begin{aligned} & (2e^{r\tau} S \gamma (r - \mu)^2 + 4e^{2r\tau} S^2 \gamma^2 (r - \mu) \sigma^2 (H - EP_s) \\ & + 2e^{3r\tau} S^3 \gamma^3 \sigma^4 (H - EP_s)^2 - \sigma^2 (g_H^1)^2) + O(\sqrt{\epsilon}) = 0 \end{aligned} \quad (3.4.2)$$

This gives

$$g^1(t, H, S) = \sqrt{\frac{\sigma^2}{2}} (e^{r\tau} S \gamma)^{\frac{3}{2}} (H - H^*)^2 + f(t, S) \quad (3.4.3)$$

where H^* is the holding in the classical problem i.e. $H^* = \frac{(\mu - r)e^{-r\tau}}{\gamma S \sigma^2} + EP_s(t, S)$

Using (3.4.1) with the first correction term (3.4.3) substituted into (3.3.6) we get the optimal control to first order,

$$h_t = -\sqrt{\frac{S\gamma\sigma^2 e^{r\tau}}{2\epsilon}} (H - H^*) \quad (3.4.4)$$

3.5 Numerical Results

Taking $\mu = r = 0$ (3.3.7) becomes

$$\frac{1}{2} S^2 \sigma^2 F_{SS} + F_t = \frac{(F_H + \gamma SF)^2}{4\epsilon \gamma SF} \quad (3.5.1)$$

and (3.4.4) becomes

$$h_t = -\sqrt{\frac{S\gamma\sigma^2}{2\epsilon}} (H - EP_s(t, S)) \quad (3.5.2)$$

In this section we numerically solve the pde (3.5.1) with the terminal boundary condition $F(T, H, S, -1) = -\exp(-\gamma(HS - EP(T, S)))$. (3.5.1) is a nonlinear

partial differential equation with 2 space variables and time so it is difficult to solve. We use a mixed technique to solve (3.5.1). For the LHS of (3.5.1) we use a Crank- Nicolson scheme (with a 3 point central difference scheme for the second derivative in the S direction) and for the RHS of (3.5.1) we use an explicit scheme. This has the advantage that for each S - H grid at each timestep we are solving a series of 1-dimensional problems. Figures (3-1) to Figures (3-7) were produced with H ranging from -1 to 1 with a step size, $\Delta_H = 2/50$, and the values of S for which we calculate the value function ranging from $S_{\min} = 300/51$ to $S_{\max} = 300$ with a step size, $\Delta_S = 300/51$, with the boundary condition that the value function equals the value function in the classical case imposed at $S_{\max} + \Delta_S$ and at $S_{\min} - \Delta_S$ i.e. we assume that at $S = 300 + \Delta_S$ and $S = 300/51 - \Delta_S = 0$ the value function is $-\exp(-\gamma (H(300+\Delta_S)-EP(t, 300+\Delta_S)))$ and $-\exp(\gamma EP(t, 0))$ respectively.

In the H direction, when calculating the first derivative we use a 5 point central difference scheme(the central point having zero weight) and 6 points close to the boundary where we cannot use the 5 point central difference scheme i.e. (supressing dependence on other variables except H and S) the central difference approximation is

$$F_H(H, S) = \frac{1}{\Delta_H} \left(\frac{1}{12} F(H - 2\Delta_H, S) - \frac{2}{3} F(H - \Delta_H, S) + 0 F(H, S) + \frac{2}{3} F(H + \Delta_H, S) - \frac{1}{12} F(H + 2\Delta_H, S) \right) + O(\Delta_H^4) \quad (3.5.3)$$

and the two approximations close to the boundary $H = -1$ are

$$F_H(H, S) = \frac{1}{\Delta_H} \left(-\frac{137}{60} F(H, S) + 5 F(H + \Delta_H, S) + -5 F(H + 2\Delta_H, S) + \frac{10}{3} F(H + 3\Delta_H, S) - \frac{5}{4} F(H + 4\Delta_H, S) + \frac{1}{5} F(H + 5\Delta_H, S) \right) + O(\Delta_H^5) \quad (3.5.4)$$

and

$$\begin{aligned}
F_H(H, S) = & \frac{1}{\Delta_H} \left(-\frac{1}{5} F(H - \Delta_H, S) - \frac{13}{12} F(H, S) + 2 F(H + \Delta_H, S) \right. \\
& \left. - 1 F(H + 2\Delta_H, S) + \frac{1}{3} F(H + 3\Delta_H, S) - \frac{1}{20} F(H + 4\Delta_H, S) \right) + O(\Delta_H^5)
\end{aligned} \tag{3.5.5}$$

and we have similar expressions close to the boundary $H = 1$.

The parameter values we use to produce figures (3-1) to figures (3-7) are $\sigma = 0.2, \gamma = 0.01, \varepsilon = 0.001, T = 5, K = 50$ where T is the time to expiry and K is the strike of the option. The number of timesteps we use are 40,000. We calculate the optimal control from (3.3.6) using the value function obtained at time 0. Comparing the optimal control from the numerical procedure with the asymptotic control at time 0, figures (3-1) and (3-2) respectively, we have good agreement except for high values of the share price where the boundary condition affects the control from the numerical procedure. We also run the numerical procedure using (3.5.2) as the control. Figure (3-3) shows the comparison of the value functions labelled $V_{optimal}$ and $V_{asymptotic}$ for the value function using the optimal control and the value function using the asymptotic control respectively. Figure (3-3) shows that the worst agreement between the value functions is at high values of the share price towards the extremes of the shareholding i.e. $H=-1$ and $H=1$ which is not surprising considering that is where the asymptotic and optimal control differ the most.

To get a clearer picture of the optimal control from the numerical procedure, figure (3-1), and the asymptotic control, figure (3-2), we plot them as a function of S with fixed H . Figures (3-4) and (3-5) show the variation in the optimal control with the share price with a fixed value of $H = -1$ and $H = -0.48$ respectively. We have good agreement until high values of S where the optimal control rapidly decreases (due to the upper boundary condition on the share price). Figures (3-6) and (3-7) show the variation in the optimal control with the share price with a fixed value of $H = 0.48$ and $H = 1$ respectively. We have good agreement until high values of S where the optimal control rapidly increases (due to the upper boundary condition on the share price). The behaviour of the control at the upper boundary is as expected since we have imposed the liquid market

boundary condition there it seems sensible that you would decrease the magnitude of the control to reduce liquidity costs (similarly for the lower boundary). As S approaches 0 we see that the \sqrt{S} dependence in (3.5.2) becomes dominant and sends the asymptotic control to zero.

The effect on the control at the upper S boundary due to imposing the liquid value function there is unwanted so we re-do the numerical routine using 'natural' boundary conditions on the maximum (and minimum values) for S . This means we use a 1 sided approximation for the second derivate of the value function with respect to S . Doing a Taylor expansion yields

$$F_{SS}(H, S) = \frac{1}{\Delta_S^2} \left(\frac{35}{12} F(H, S) - \frac{26}{3} F(H, S - \Delta_S) + \frac{19}{2} F(H, S - 2\Delta_S) - \frac{14}{3} F(H, S - 3\Delta_S) + \frac{11}{12} F(H, S - 4\Delta_S) \right) + O(\Delta_S^3) \quad (3.5.6)$$

and

$$F_{SS}(H, S) = \frac{1}{\Delta_S^2} \left(\frac{35}{12} F(H, S) - \frac{26}{3} F(H, S + \Delta_S) + \frac{19}{2} F(H, S + 2\Delta_S) - \frac{14}{3} F(H, S + 3\Delta_S) + \frac{11}{12} F(H, S + 4\Delta_S) \right) + O(\Delta_S^3) \quad (3.5.7)$$

which we use at the upper and lower boundary respectively. We use 200,000 timesteps, other parameters as for figures (3-1) to figures (3-7), and figure (3-8) shows the comparison with the asymptotic control. We see that we no longer have the unwanted effect at the upper boundary we saw with the previous boundary conditions for S . Figures (3-9) to (3-12) show the difference between the optimal controls at time 0 (for both boundary conditions) minus the asymptotic controls. Since the difference using the boundary conditions from the liquid value function swamps that of using natural boundary conditions we show figure (3-10) with the difference only using natural boundary conditions in figure (3-13) for $H = -0.48$.

In the rest of the numerical examples we just use the natural boundary conditions since they give better results. Also, we no longer need to go so high in the S direction since we are not imposing artificial boundary conditions there. Figure (3-14) shows the optimal control and asymptotic control where the parameters

are $K = 15$ and $S_{\min} = 30/101$ and $S_{\max} = 30$ with $\Delta_S = 30/101$ using 100,000 timesteps (the other parameters the same as those used to produce figures (3-1) to Figures (3-7), n and m in the caption refer to the number of points in the S and H directions respectively). We also show in figure (3-15) the difference between the optimal control and the asymptotic control at time 0 since the difference is small and hard to see in (3-14). Figure (3-16) shows the effect on the control of changing the number of timesteps from 50,000 to 100,000 on the control at $H = -0.48$. The effect is small, the order of 10^{-5} . Figure (3-17) shows the control at $H = -0.48$ for two values of γ , $\gamma = 0.01$ and $\gamma = 0.02$, other parameters as for (3-14).

We now calculate option prices. We use a modified version of (3.1.5) to calculate the option price. The option price p is given by

$$V(t, 0, Y + p, S, -1) = V(t, 0, Y, S, 0) \quad (3.5.8)$$

Using (3.3.1) this gives

$$\exp(-\gamma p)F(t, 0, S, -1) = F(t, 0, S, 0) \quad (3.5.9)$$

so that p is

$$p = -1/\gamma \log \left(\frac{F(t, 0, S, 0)}{F(t, 0, S, -1)} \right) \quad (3.5.10)$$

Figure (3-18) shows the difference between the option price in the illiquid market and the Black Scholes price as a function of share price with $T = 4$, $\varepsilon = 1$ and $\gamma = 0.05$. We lift the upper boundary in the S direction slightly to $S_{\max} = 141 * 30/101$ but keep the same step size and use 160,000 timesteps, the other parameters are the same as to produce (3-14). The largest difference seems to be slightly below the strike price, $K = 15$, with the difference tending to zero at the extremes of the share price. Figure (3-19) explores the effect of varying the volatility. It plots the price difference for two values of σ , $\sigma = 0.1$ and $\sigma = 0.05$, $S_{\max} = 30$ and $\Delta S = 30/101$. All other parameters are the same as for figure (3-18). Figure (3-20) shows the option price difference as a function of time, plotted every 0.25 years out to 5 years, with $S = 50 * 30/101$. Each 0.25 years was computed with 10,000 timesteps, other parameters are as for the

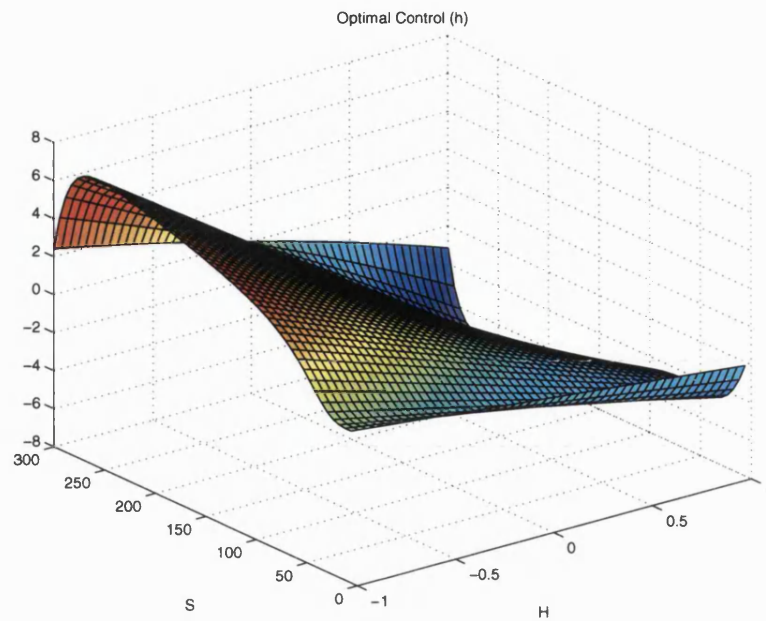


Figure 3-1: Optimal Control

computation of figure (3-18). Figures (3-21) to (3-23) explore the change in option price difference for the case $\sigma = 0.1$ in figure (3-19) with changing the domain size. Figure (3-21) shows the change in option price difference due to increasing the timesteps from 80,000 to 160,000, figure (3-22) the effect of doubling the number of steps in the H and S direction with the number of timesteps fixed at 160,000 and figure (3-23) shows the effect of increasing the timesteps to 320,000 with the same space resolution as for figure (3-22). The effect on the option price difference is small so we can have faith in the results.

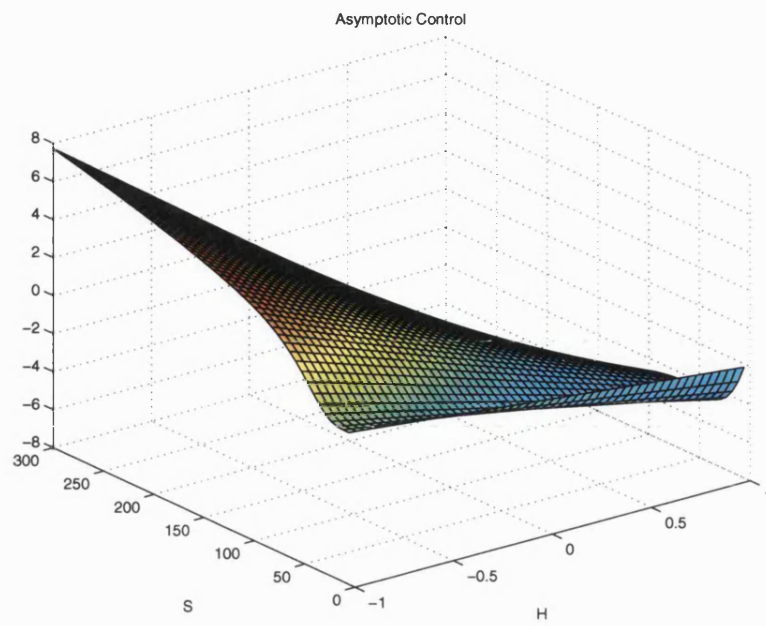


Figure 3-2: Asymptotic Control

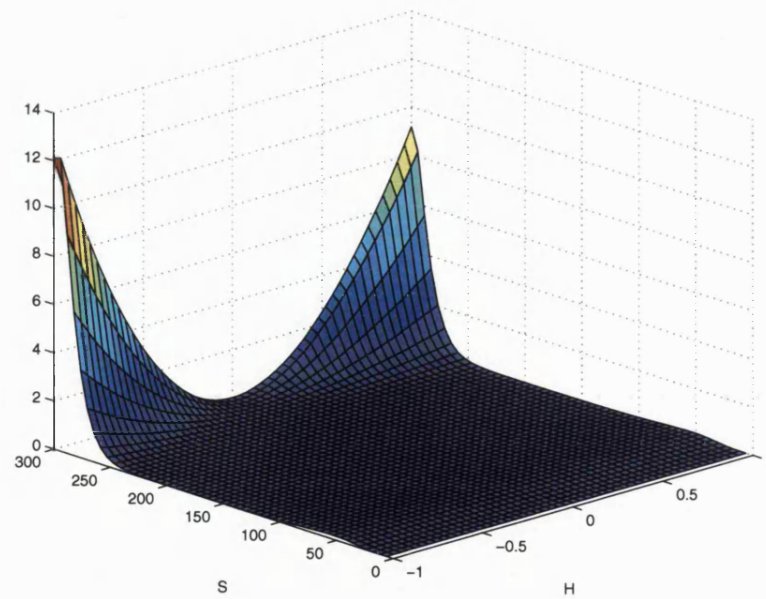


Figure 3-3: $-(V_{Optimal} - V_{Asymptotic})/V_{Optimal} * 10000$

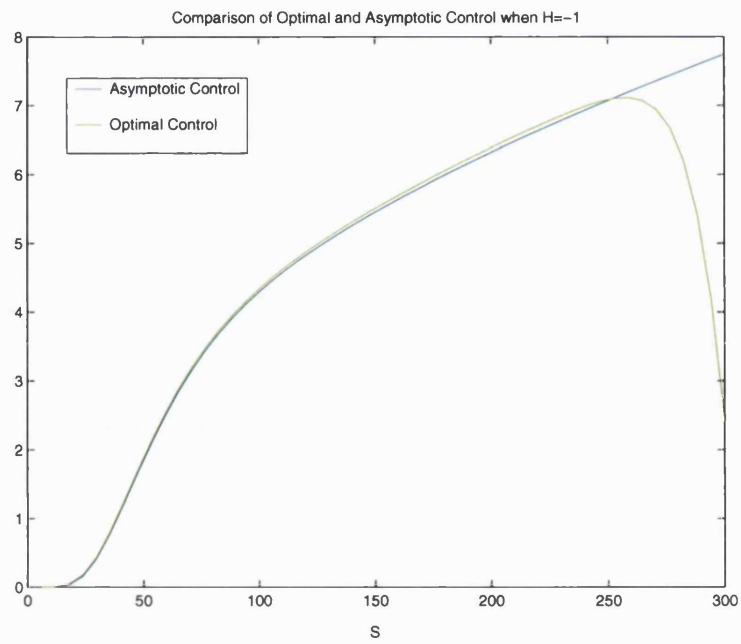


Figure 3-4: Optimal and Asymptotic Control, $H=-1$

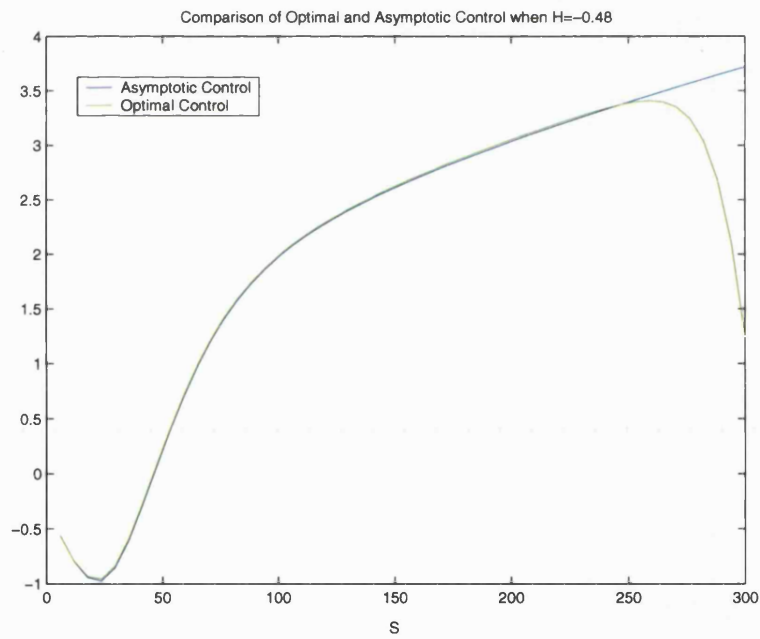


Figure 3-5: Optimal and Asymptotic Control, $H=-0.48$

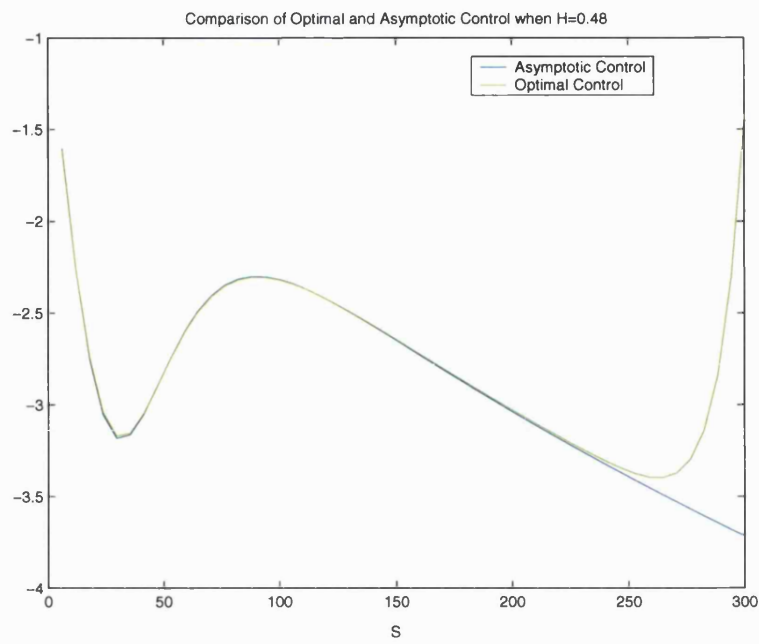


Figure 3-6: Optimal and Asymptotic Control, $H=0.48$

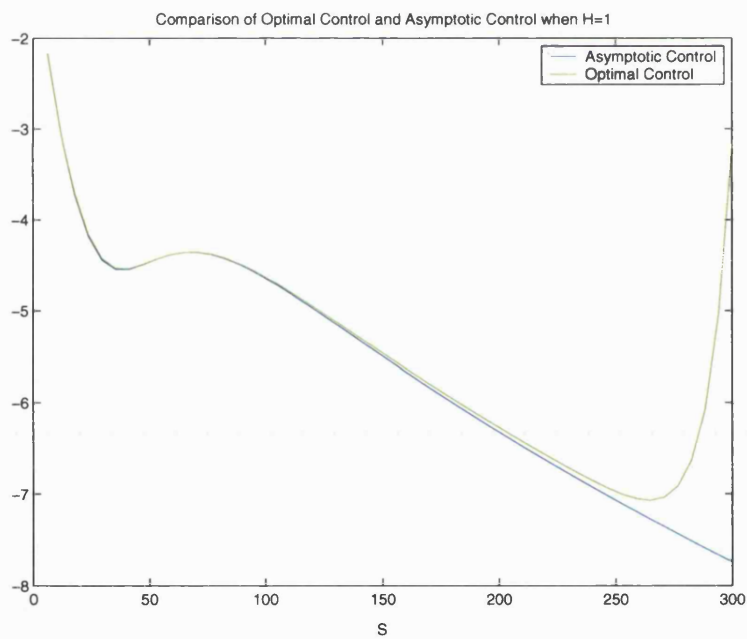


Figure 3-7: Optimal and Asymptotic Control $H=1$

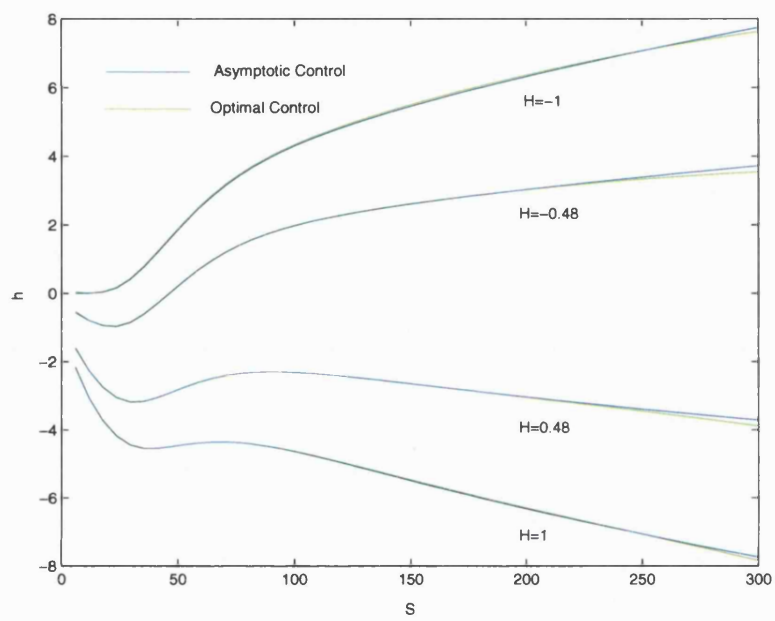


Figure 3-8: Optimal and Asymptotic Control using natural boundary conditions

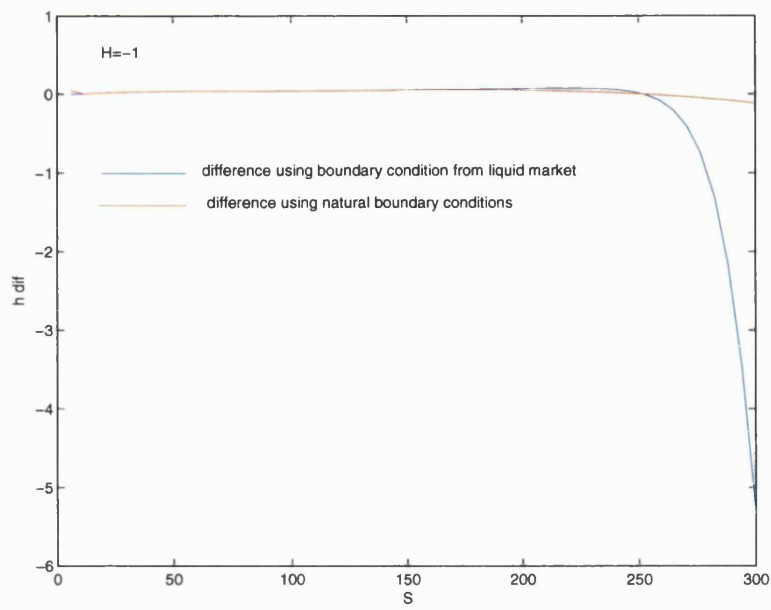


Figure 3-9: Difference between optimal control and asymptotic control for both S boundary conditions

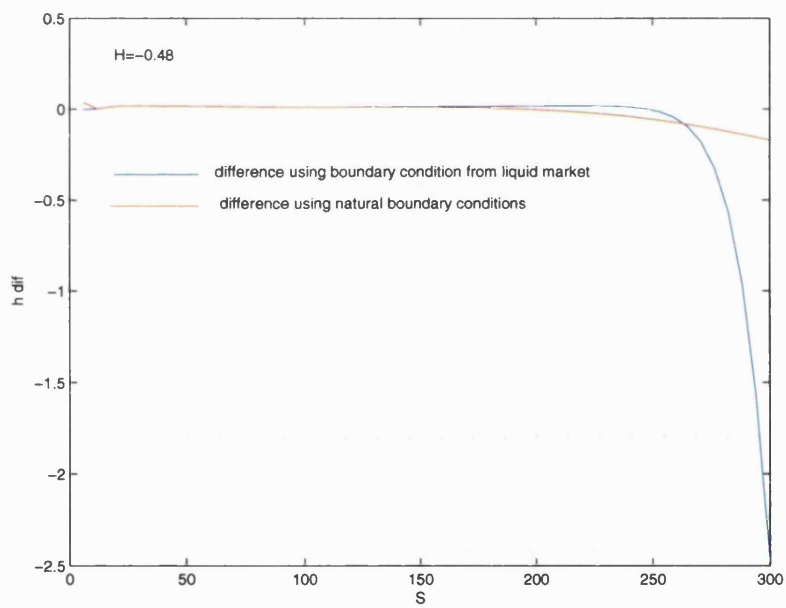


Figure 3-10: Difference between optimal control and asymptotic control for both S boundary conditions

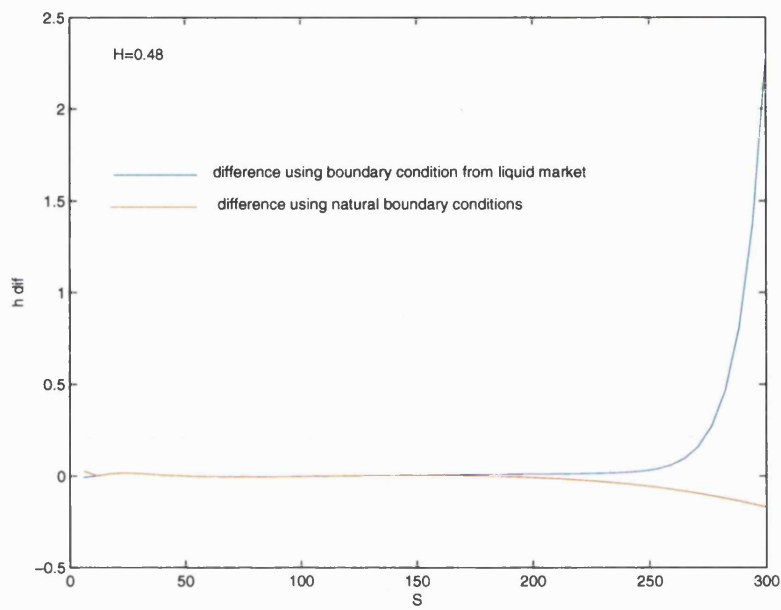


Figure 3-11: Difference between optimal control and asymptotic control for both S boundary conditions

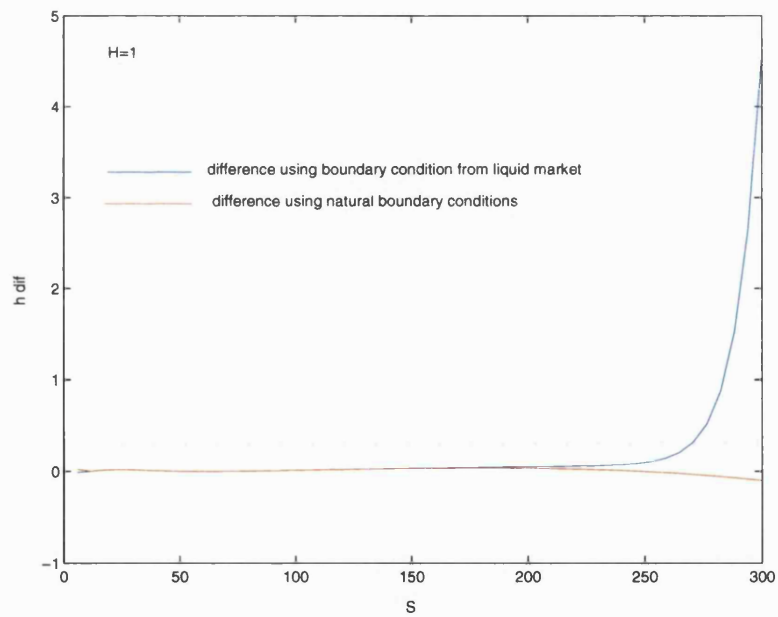


Figure 3-12: Difference between optimal control and asymptotic control for both S boundary conditions

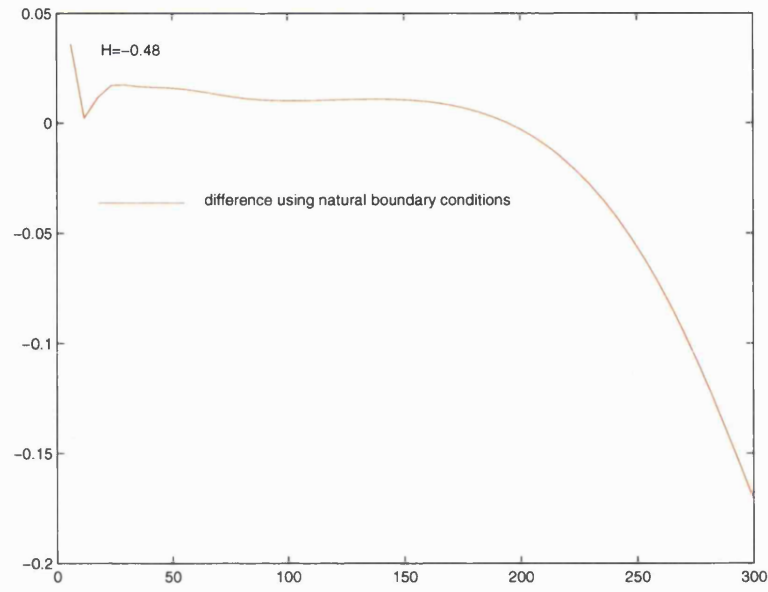


Figure 3-13: Difference between optimal control and asymptotic control for natural boundary conditions

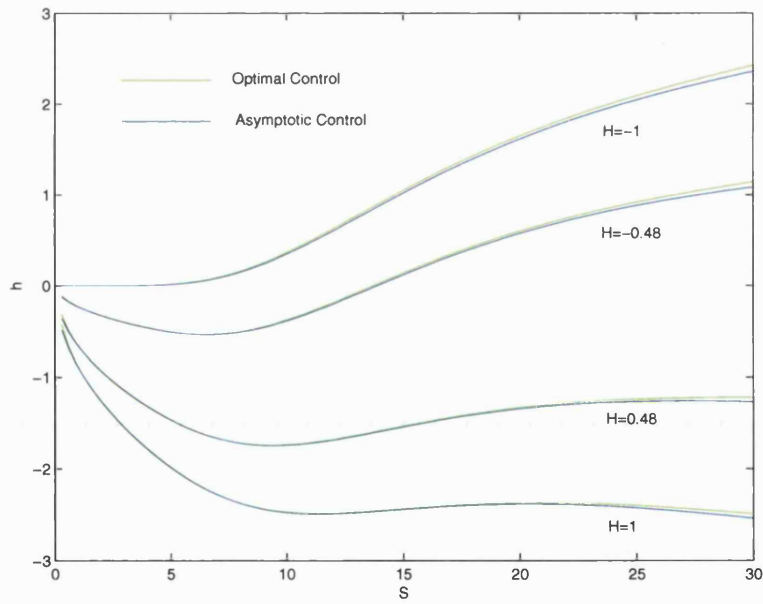


Figure 3-14: Optimal control and Asymptotic control $K = 15$, $T = 5$, 100,000 timesteps, $\sigma = 0.2$, $\gamma = 0.01$, $\varepsilon = 0.001$, $n=101$, $m=51$

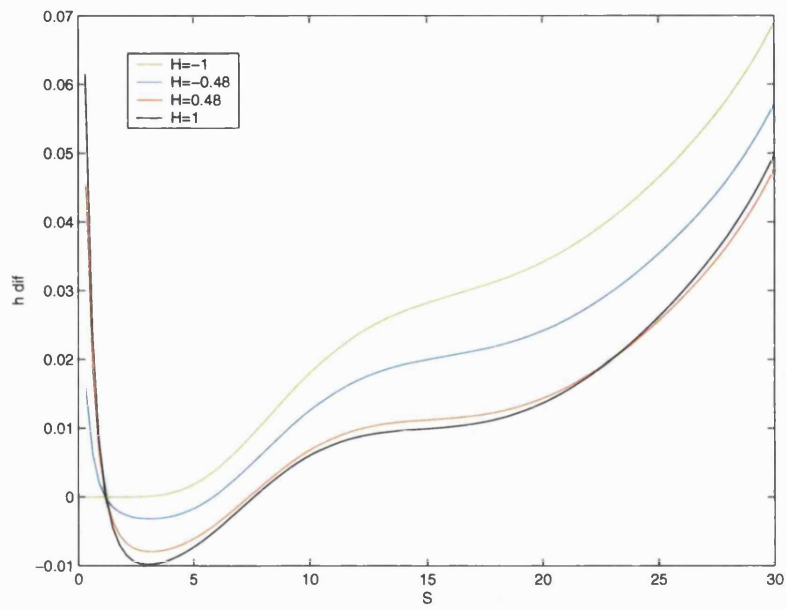


Figure 3-15: Optimal control-Asymptotic control

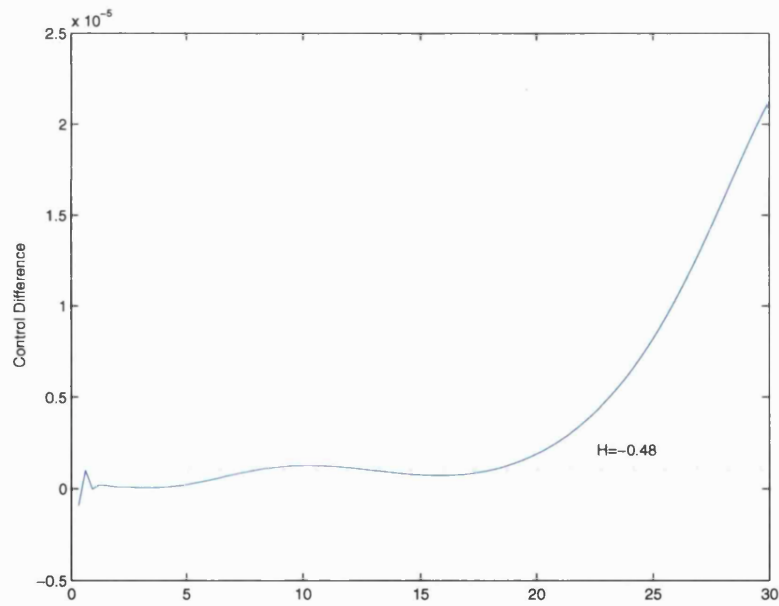


Figure 3-16: Optimal control difference due to changing the number of timesteps, $K = 15$, $T = 5$, 100,000 timesteps case minus 50,000 timesteps case, $\sigma = 0.2$, $\gamma = 0.01$, $\varepsilon = 0.001$, $n=101$, $m=51$

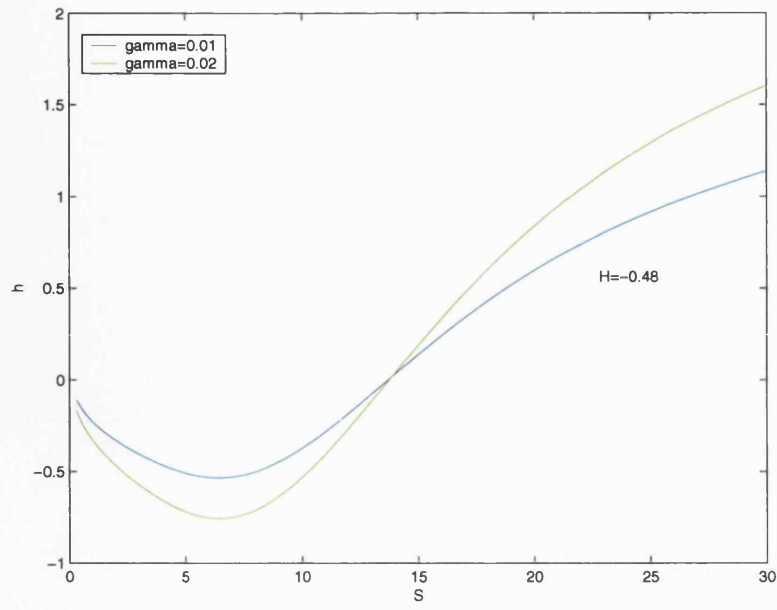


Figure 3-17: effect of γ on h

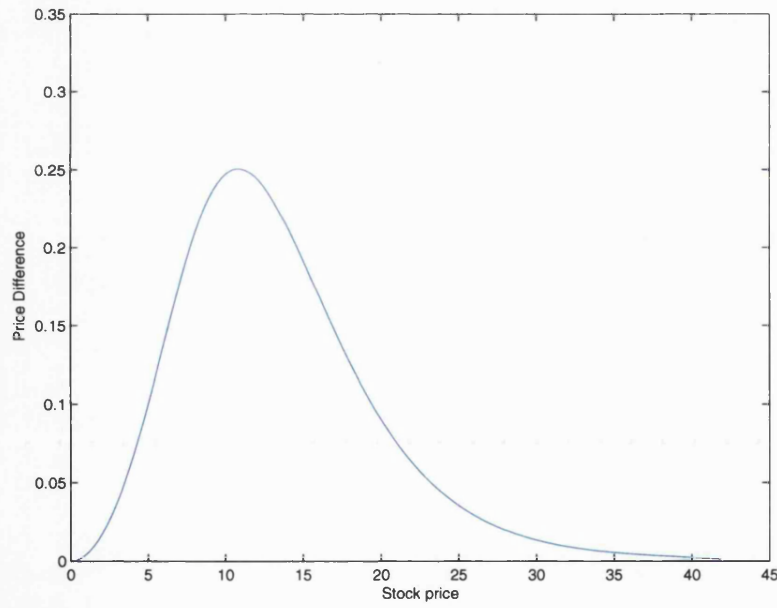


Figure 3-18: Illiquid Option Price minus Black Scholes Price $K = 15$, $T = 4$, $\sigma = 0.2$, 160,000 timesteps, $\varepsilon = 1$, $\gamma = 0.05$, $n=141$, $m=51$, $S_{\max} = 30/101 * 141$

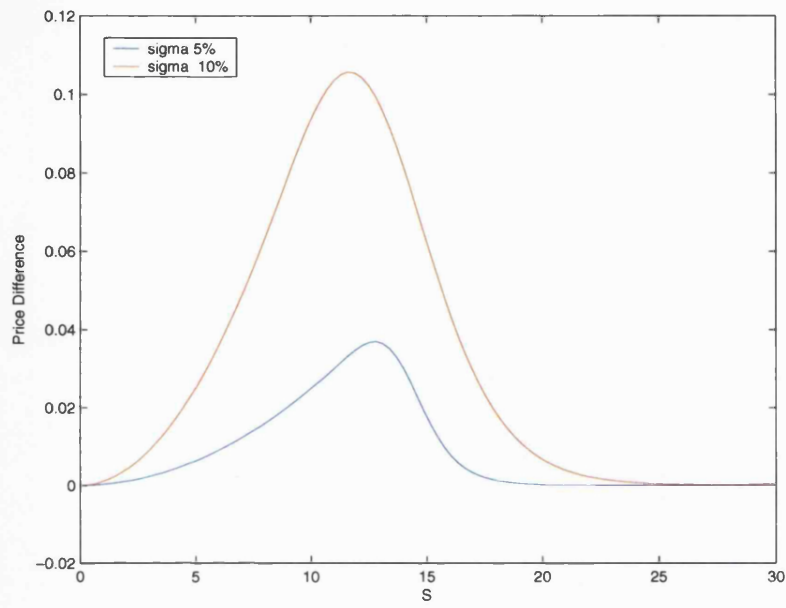


Figure 3-19: Illiquid Option Price minus Black Scholes Price $K = 15$, $T = 4$, $\sigma = 0.1$ and $\sigma = 0.05$, 160,000 timesteps, $\varepsilon = 1$, $\gamma = 0.05$, $n=101$, $m=51$ and $S_{\max} = 30$

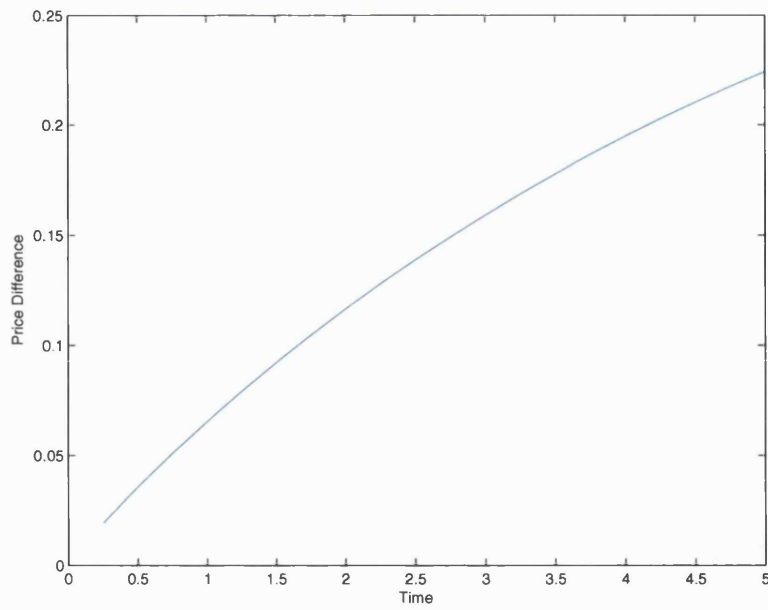


Figure 3-20: Illiquid Option Price minus Black Scholes Price, $K = 15$, $\sigma = 0.2$, 200,000 timesteps, $\varepsilon = 1$, $\gamma = 0.05$, $n=141$, $m=51$ at $S = 30/101 * 50$

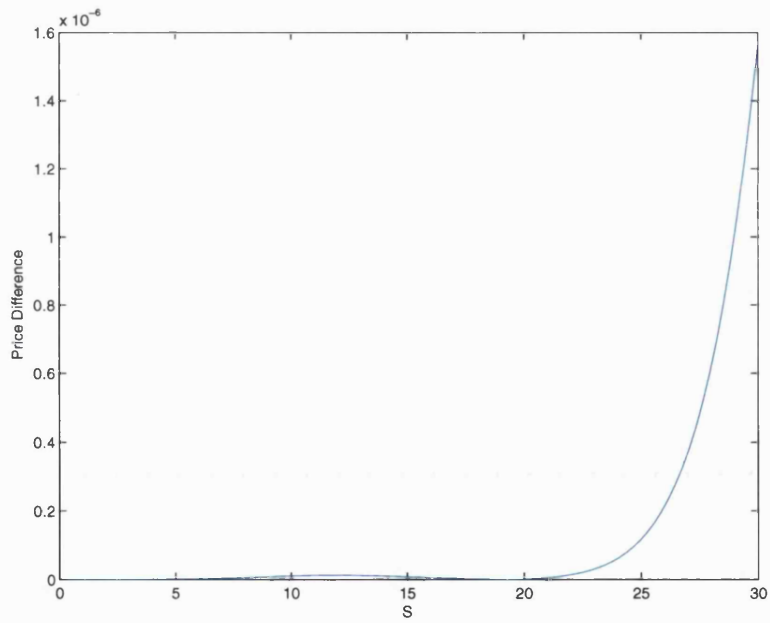


Figure 3-21: Illiquid Option Price minus Black Scholes Price difference due to increasing number of timesteps from 80,000 to 160,000, $K = 15$, $T = 4$, $\sigma = 0.1$, $\varepsilon = 1$, $\gamma = 0.05$, $n=101$, $m=51$, $S_{\max} = 30$

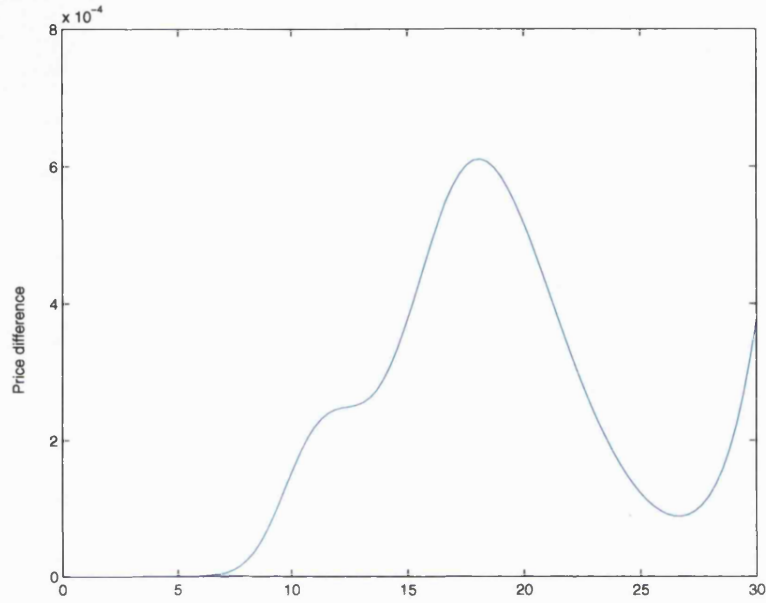


Figure 3-22: Illiquid Option Price minus Black Scholes Price difference due to increasing space resolution, $K = 15$, $T = 4$, $\sigma = 0.1$, 160,000 timesteps $\varepsilon = 1$, $\gamma = 0.05$, (m=101 and n=202) case minus (m=51, n=101) case, difference plotted for common points, $S_{\max} = 30$

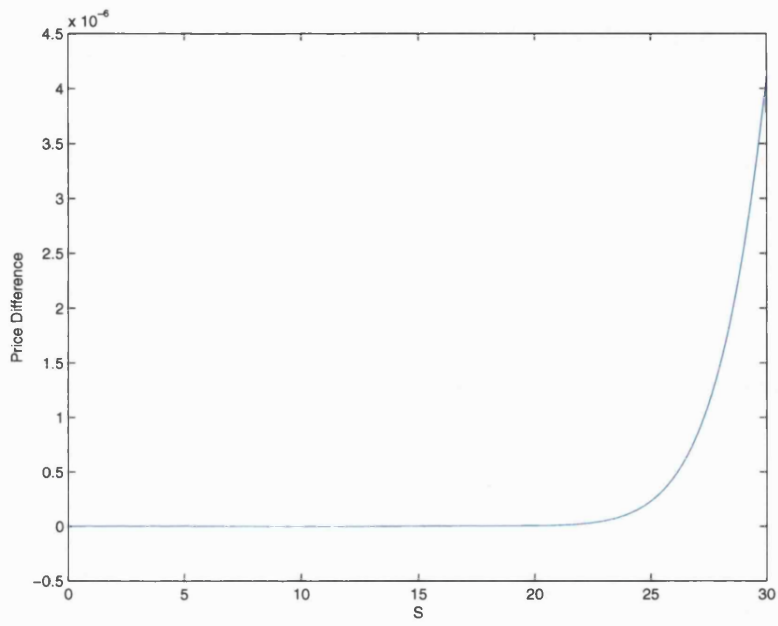


Figure 3-23: Illiquid Option Price minus Black Scholes Price difference due to increasing number of timesteps from 160,000 to 320,000, $K = 15$, $T = 4$, $\sigma = 0.1$, $\varepsilon = 1$, $\gamma = 0.05$, $n=202$, $m=101$, $S_{\max} = 30$

3.6 Conclusions

In this chapter we have applied the modified dynamics developed in chapter 1 for an illiquid market to the problem of pricing a put option (from the writer's point of view). Since the market is no longer complete the payoff of a put option cannot be replicated as in the classical Black-Scholes framework. There is no unique way to price an option in this market. We take the approach of utility indifference pricing where two utility maximisation problems are looked at. One is the maximisation of terminal utility of wealth with the option liability and the other, without. These two values imply the price of the option.

In the classical complete market case the technique gives the hedge as the delta of the put option. However in our problem we are not controlling the share holding, rather h_t , the rate of change of the share holding with respect to time. The asymptotic control given by (3.4.4) tells us that h_t is proportional to number of shares away from the optimal holding in the classical case we are, and this control agrees well with the control found from the numerical procedure.

Chapter 4

Monte Carlo Valuation of American Options using the Dual method

4.1 Introduction

An American option contract allows the holder to exercise any time before expiry, T , in contrast to a European option which can only be exercised at expiry. Mathematically we can formulate the problem as follows. Let X_t be a stochastic process. Suppose the payoff on exercise of the option at time t is $\zeta_t = \zeta(X_t)$ and the discounted payoff is $Z_t = \exp(-\int_0^t r_s ds)\zeta_t$ where r_s is the interest rate process. The price of a an American option is

$$\sup_{0 \leq \tau \leq T} \mathbb{E}Z_\tau \tag{4.1.1}$$

where τ is a stopping time.

A popular technique is to find the price on a lattice using dynamic programming e.g. using a binomial tree the value of the American option is determined recursively. If we discretize time into N steps and let X_i be the state of the markov chain at time t_i and let $V_i(x)$ be the option price at time t_i when $X_i = x$

then

$$\begin{aligned} V_N(x) &= \zeta(x) \\ V_i(x) &= \max(\zeta(x), \mathbb{E}[\gamma_i V_{i+1}(X_{i+1}) | X_i = x]) \end{aligned} \quad (4.1.2)$$

where $i = 0, 1, \dots, N - 1$, and γ_i is the discount factor between t_i and t_{i+1} .

Alternatively the pde formulation can be used. If we consider an American option on a single asset following a geometric Brownian motion, S_t , the pde formulation for the price, $V(t, S)$, is

$$\begin{aligned} \mathcal{L}V &= \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV + V_t \leq 0 \\ \zeta - V &\leq 0 \\ \mathcal{L}V(V - \zeta) &= 0 \end{aligned} \quad (4.1.3)$$

together with the condition, $V(T, S) = \zeta(S)$. This can be solved using a finite difference scheme. Backward in time techniques such as dynamic programming and finite difference methods are fine for low dimensional problems but because the computational effort grows exponentially with the number of state variables they are not practical for high dimensional problems. Monte Carlo simulation on the other hand grow approximately linearly with the number of state variables which makes it more appealing for high dimensional problems. Monte Carlo, however, does not lend itself easily to American style options. If the optimal stopping time τ was known then the valuation would be easy but unfortunately this is not the case.

Since Tilley (1993) various attempts to price American options using Monte Carlo methods have emerged. Broadie and Glasserman (1997) provide a technique which produces two estimates, one biased high and another biased low (they are actually considering a Bermudan option which has a fixed number of exercise dates). They simulate a random tree characterised by the number of exercise opportunities, m , and the number of branches per exercise opportunity, b . To obtain the high estimate they apply a standard dynamic programming technique to the tree. Intuitively the high bias is due the algorithm looking into the future to arrive at the option value. The key to removing the bias is to separate the paths used to decide whether to exercise from the paths to calculate

the continuation value. This results in a low bias. The intuitive reason for this is that you are effectively taking the expected value of a sub-optimal exercise strategy. The main disadvantage with this technique is that you have b^m nodes at expiry so it is not feasible for an option with a large number of exercise dates. Broadie and Glasserman (2004) introduce a stochastic mesh method. Here the number of nodes per exercise opportunity is fixed. When calculating expected values, transitions are allowed from every node at each exercise opportunity to every node at the next exercise opportunity hence the procedure produces a mesh rather than a tree.

Barraquand and Martineau (1995) use a technique of partitioning the state space at each exercise opportunity and calculating a strategy and value which is constant on each partition. Monte Carlo simulation is used to estimate the probability of going from a partition at one time period to another partition in the next time period.

Tsitsiklis and Van Roy (2001) and Longstaff and Schwartz (2001) use a regression technique to estimate the continuation value. Longstaff and Schwartz (2001) approximate the continuation value using $\mathbb{E}[V_{i+1}(X_{i+1})|X_i = x] = \sum_{k=1}^{k=M} \beta_{ik} \psi_k(x)$ where the ψ_k are basis functions. To get the values of the β_{ik} they do a least squares regression on the values at the next time step. They only consider nodes which are in the money. These approaches all try to approximately solve (4.1.1) in some way.

There is an alternative approach introduced independently by Haugh and Kogan (2001) and Rogers (2002). Here we present the theory following Rogers (2002). Using the notation of (4.1.1) under some integrability assumptions it is well known that the process

$$Y_t^* = \sup_{t \leq \tau \leq T} \mathbb{E}[Z_\tau | \mathcal{F}_t] \quad (4.1.4)$$

is a supermartingale and admits a Doob-Meyer decomposition

$$Y_t^* = Y_0^* + M_t^* - A_t^* \quad (4.1.5)$$

where M_t^* is a martingale vanishing at zero and A_t^* is an increasing process also vanishing at zero. Let H be the space of uniformly integrable martingales with

$M_0 = 0$. Then for any $M \in H$

$$\begin{aligned}
Y_0^* &= \sup_{0 \leq \tau \leq T} \mathbb{E} Z_\tau \\
&= \sup_{0 \leq \tau \leq T} \mathbb{E}[Z_\tau - M_\tau] \\
&\leq \mathbb{E} \left[\sup_{0 \leq t \leq T} (Z_t - M_t) \right]
\end{aligned} \tag{4.1.6}$$

and to show that equality is achieved when $M_t = M_t^*$

$$\begin{aligned}
\inf_{M \in H} \mathbb{E} \left[\sup_{0 \leq t \leq T} (Z_t - M_t) \right] &\leq \mathbb{E} \left[\sup_{0 \leq t \leq T} (Z_t - M_t^*) \right] \\
&\leq \mathbb{E} \left[\sup_{0 \leq t \leq T} (Y_t^* - M_t^*) \right] \\
&= \mathbb{E} \left[\sup_{0 \leq t \leq T} (Y_0^* - A_t^*) \right] \\
&= Y_0^*
\end{aligned} \tag{4.1.7}$$

so we have

$$Y_0^* = \mathbb{E} \left[\sup_{0 \leq t \leq T} (Z_t - M_t^*) \right] \tag{4.1.8}$$

(4.1.8) represents the dual method of pricing using Monte Carlo. The pricing technique involves finding a martingale, M_t , with $M_0 = 0$ and evaluating the expression $\mathbb{E}[\sup_{0 \leq t \leq T} (Z_t - M_t)]$ which from the previous theory gives an upper bound on the price, the exact price only being achieved when $M_t = M_t^*$. Haugh and Kogan (2001) and Anderson and Broadie (2004) obtain upper bounds using this technique from lower bounds. After calculating an approximation to the option price using a primal method Haugh and Kogan (2001) use the change in the option price minus the expected value of that change as the martingale increments. Anderson and Broadie (2004) generate their martingales from stopping rules.

Rogers (2002) takes a different approach in that only the dual problem is considered and martingales are constructed on a problem specific basis. He finds that you can get reasonably close prices by judicious choice of martingale. If we set $\eta_t = \mathbb{E}[\sup_{0 \leq t \leq T} (Z_t - M_t) | \mathcal{F}_t]$ then the random variable η_0 is a constant if the martingale M_t^* is chosen. Rogers (2002) explains that M_t has an interesting interpretation in terms of hedging. If you write the option for η_0 and suppose that M_t is the discounted gains from trade then using $Z_t \leq \eta_T + M_t$ and taking

conditional expectations we get

$$Z_t \leq \mathbb{E}[\eta_T - \eta_0 | \mathcal{F}_t] + M_t + \eta_0 \quad (4.1.9)$$

so that if the option is exercised at time t the shortfall is at worst $\mathbb{E}[(\eta_T - \eta_0) | \mathcal{F}_t]^+$, the mean of which is bounded by $\mathbb{E}[|(\eta_T - \eta_0)|]$ so it is desirable to have this quantity small (in the case we choose M_t^* as the discounted gains from trade process it is zero). This quantity is shown in the tables as the mean absolute deviation (MAD). Apart from Rogers (2002), other studies have not focused on the problem of getting better hedges in the sense that the MAD is small, they just look at pricing. Rogers (2002) in the conclusion states that in the few examples he studies, that it is easy to get reasonable prices but the large MAD values from the hedging policies is so large that calling them hedging policies is a bit of a misnomer!

We look at the case of an American put on a single asset and an American min-put on two assets. We refine the martingales used by Rogers (2002) to get lower values for the MAD. Lamper and Howison (2003) also look at the dual method of pricing. They consider adding binary puts and calls in addition to the discounted put as martingales.

4.2 Using the discounted put martingale

The process Z in this section and section (4.3) is given by

$$Z_t = e^{-rt}(K - S_t)^+ \quad (4.2.1)$$

where

$$S_t = S_0 \exp(\sigma W_t + (r - q - \sigma^2/2)t), \quad (4.2.2)$$

where W_t is a standard brownian motion, r is the risk free rate, q is the dividend yield (which we will take as zero) and K is the strike of the option.

To produce uniform random numbers $\in (0, 1)$ the routine ran2 in Press, Teuolsky, Vetterling and Flannery (2002) was used. To get normal deviates from random deviates uniformly distributed in $(0, 1)$ we use the box-muller transform method. This relies on the following transformation of two uniformly deviates on

$(0, 1)$, x_1 and x_2 .

$$\begin{aligned} y_1 &= \sqrt{-2 \log x_1} \cos(2\pi x_2) \\ y_2 &= \sqrt{-2 \log x_1} \sin(2\pi x_2) \end{aligned}$$

The density function of (y_1, y_2) is given by $|\frac{\partial x_1}{\partial y_1} \frac{\partial x_2}{\partial y_2} - \frac{\partial x_1}{\partial y_2} \frac{\partial x_2}{\partial y_1}| = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_1^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_2^2}$ so that y_1 and y_2 are independently normally distributed with mean 0 and variance 1. The algorithm used taken from Press et al. (2002) actually takes random points in the unit circle (v_1, v_2) with $x_1^2 = v_1^2 + v_2^2$ and $2\pi x_2$ is the angle made by the line from the origin to (v_1, v_2) with the x-axis so that $\cos(2\pi x_2) = \frac{v_1}{x_1}$ and $\sin(2\pi x_2) = \frac{v_2}{x_1}$.

Let T be the time to expiry, $\Delta t = \frac{T}{N}$ and $t_i = i\Delta t$ for $i \in \{0, \dots, N\}$. We simulate the asset process (4.2.2) recursively as follows:

$$S_{t_{i+1}} = S_{t_i} \exp\left((r - q - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t} y_{i+1}\right) \quad (4.2.3)$$

where y_1, y_2, \dots, y_N are independent standard normals obtained from the procedure outlined earlier and $Z_{t_i} = e^{-rt_i}(K - S_{t_i})^+$. M_{t_i} is a martingale based on the simulated asset path up to time t_i . In order to calculate

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} (Z_t - M_t)\right] \quad (4.2.4)$$

we use a Richardson extrapolation of convergence order 1 with 50 and 25 timesteps. For the j th simulated path of $Z_t - M_t$ using $N = 50$ timesteps we record $e_1^j = \max_{i \in \{0, 1, \dots, N\}} (Z_{t_i} - M_{t_i})$ and record $e_2^j = \max_{i \in \{0, 1, \dots, \frac{N}{2}\}} (Z_{t_{2i}} - M_{t_{2i}})$. To get our estimate of (4.2.4), call it $\tilde{\eta}_0$, we calculate $\tilde{\eta}_0 = \frac{\sum_{j=1}^X (2e_1^j - e_2^j)}{X}$ where X is the number of paths in the simulation. We then calculate the MAD as $\frac{\sum_{j=1}^X |2e_1^j - e_2^j - \tilde{\eta}_0|}{X}$. The code was written in c++ and run on a 2.67 GHz machine with 512MB of RAM. We calculate results based on the martingale given by the discounted put price, $e^{-rt}EP(t, S_t)$ where $EP(t, S_t)$ is the price of a European put at time t on an asset with price S_t . Recall

$$\begin{aligned} e^{-rt}EP(t, S_t) &= e^{-rt} \mathbb{E}[e^{-r(T-t)}(K - S_T)^+ | \mathcal{F}_t] \\ &= \mathbb{E}[e^{-rT}(K - S_T)^+ | \mathcal{F}_t] \end{aligned} \quad (4.2.5)$$

which is a martingale.

The code to produce the results in table (4.1) and table (4.2) was programmed jointly with John Aquilina at the University of Bath.

In the Monte Carlo simulation the martingale we actually use is

$$M_{t_i} = \lambda(e^{-rt_i}EP(t_i, S_{t_i}) - EP(0, S_0)) \quad (4.2.6)$$

Multiplying a martingale starting at 0 by a factor, λ , still is a martingale starting at 0 and a value of λ other than 1 may give a lower price so it makes sense to search for the optimal value of λ . We call λ the weight of the martingale. The value of λ is found through an optimisation procedure just using 500 paths for the expectation (4.2.4). We calculate (4.2.4) as explained earlier using 500 paths and the optimiser¹ searches for the best choice of λ that minimises some objective. Rogers (2002) uses the discounted put minus its initial value as a martingale with a weight obtained by having the optimisation objective to minimise the price i.e. the expectation (4.2.4). Since we are interested in getting better hedges we also minimise the sum of the price and the standard deviation as our optimisation objective. Remember if we find M_t^* the standard deviation is 0. Once the optimal λ is found the expectation (4.2.4) is found using 5000 paths.

The parameters we use are $T = 0.5$, $\sigma = 0.4$, $r = 0.06$, $K = 100$ and $q = 0$. Table (4.1) is the result of the simulation with the objective of minimising price in the optimisation procedure. In table (4.2) the objective is to minimise the standard deviation (sd) plus the price in the optimisation procedure. As noted by Rogers (2002) the values found for λ are close to 1. Table (4.2) is the benchmark case we improve on in the next section. The true American values in column 2 are from AitSahlia and Carr (1997) where they use the average obtained from a 1000 and 1001 step binomial method as the exact values.

¹We used an optimiser called cfsqp available from <http://www.aemdesign.com/>

$S(0)$	American(True)	American (MC)	Standard error	MAD	λ
80	21.6059	21.6830	0.0035	0.2101	1.0567
85	18.0374	18.0953	0.0034	0.2016	1.0510
90	14.9187	14.9607	0.0034	0.1684	1.0385
95	12.2314	12.2612	0.0030	0.1388	1.0351
100	9.9458	9.9681	0.0027	0.1134	1.0308
105	8.0281	8.0447	0.0028	0.1216	1.0234
110	6.4352	6.4488	0.0026	0.1123	1.0189
115	5.1265	5.1352	0.0019	0.0648	1.0218
120	4.0611	4.0687	0.0018	0.0658	1.0168

Table 4.1: Simulation using discounted American put as a martingale. Optimisation objective is to min (price). 5000 paths

$S(0)$	American(True)	American (MC)	Standard error	MAD	λ
80	21.6059	21.6832	0.0035	0.2108	1.0575
85	18.0374	18.0956	0.0034	0.2039	1.0523
90	14.9187	14.9605	0.0031	0.1814	1.0466
95	12.2314	12.2616	0.0028	0.1566	1.0420
100	9.9458	9.9691	0.0024	0.1254	1.0373
105	8.0281	8.0444	0.0021	0.1022	1.0339
110	6.4352	6.4472	0.0018	0.0772	1.0303
115	5.1265	5.1350	0.0016	0.0627	1.0284
120	4.0611	4.0673	0.0014	0.0489	1.0264

Table 4.2: Simulation using discounted American put as a martingale. Optimisation objective is to min (price+sd). 5000 paths

4.3 Refining the martingale

Using the discounted value of a European put option is the obvious martingale to try. The actual martingale we are looking for, M_t^* , is the martingale part of the process (4.1.4) with initial value 0. If we had an expression for the value of an American put as a function of t and S we could simply discount it and take the martingale part of it starting at 0 and this would be M_t^* . Since the value of a European put is a decreasing function of the share price, S , there is a unique value, S^* , which we can put into the European put pricing formula, $EP(t, S)$, to get the corresponding American put option price.

If we can find a good approximation $g(t, S) \approx S^*$ we can take the martingale part of $(\exp(-rt)EP(t, g(t, S)))$ and hopefully get better results. Defining $\tilde{S}_t = g(t, S_t)$. Applying Ito's formula gives

$$\begin{aligned} d(e^{-rt}EP(t, \tilde{S}_t)) &= e^{-rt}(d(EP(t, \tilde{S}_t)) - rEP(t, \tilde{S}_t)dt) \\ &= e^{-rt}\left\{ \Delta(t, \tilde{S}_t) d\tilde{S}_t + \frac{1}{2}\Gamma(t, \tilde{S}_t) d\tilde{S}_t d\tilde{S}_t + \Theta(t, \tilde{S}_t) dt - rEP(t, \tilde{S}_t)dt \right\} \end{aligned} \quad (4.3.1)$$

where Δ, Γ and Θ are greeks of the European put option. These are well known expressions.

$$\begin{aligned} \Delta(t, s) &= -e^{-q(T-t)}N(-d1(t, s)), \\ \Gamma(t, s) &= \frac{N'(d1(t, s))}{s\sigma\sqrt{T-t}}e^{-q(T-t)}, \\ \Theta(t, s) &= -\frac{sN'(d1(t, s))\sigma e^{-q(T-t)}}{2\sqrt{T-t}} - qsN(-d1(t, s))e^{-q(T-t)} + rKe^{-r(T-t)}N(-d2(t, s)) \end{aligned} \quad (4.3.2)$$

where $N(x)$ is the cumulative normal function, r is the risk free rate, K is the strike of the option, σ is the volatility of the share price process, q is the dividend yield, T is the expiry time, $d1(t, s) = \frac{\log(\frac{s}{K}) + (r - q + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$ and $d2(t, s) = d1(t, s) -$

$\sigma\sqrt{T-t}$. Substituting (4.3.2) into (4.3.1) gives

$$\begin{aligned} d(e^{-rt}EP(t, \tilde{S}_t)) &\doteq e^{-rt} \left\{ \Delta(t, \tilde{S}_t)(S_t(r-q)g_s + \frac{1}{2}S_t^2\sigma^2g_{ss} + g_t) \right. \\ &\quad + \frac{1}{2}\Gamma(t, \tilde{S}_t)\sigma^2S_t^2g_s^2 - e^{-q(T-t)}\frac{\tilde{S}_tN'(d1(t, \tilde{S}_t))\sigma}{2\sqrt{T-t}} \\ &\quad \left. + e^{-q(T-t)}(r-q)\tilde{S}_tN(-d1(t, \tilde{S}_t)) \right\} dt \end{aligned} \quad (4.3.3)$$

The symbol \doteq means two sides of the equation differ by a local martingale. If we make the substitution $g(t, S_t) = f(t, S_t)S_t$ (4.3.3) simplifies somewhat.

$$\begin{aligned} d(e^{-rt}EP(t, \tilde{S}_t)) &\doteq e^{-rt} \left\{ \Delta(t, \tilde{S}_t)((f_s + \frac{1}{2}S_tf_{ss})S_t^2\sigma^2 + S_t^2(r-q)f_s + S_tf_t) \right. \\ &\quad \left. + \frac{1}{2}\Gamma(t, \tilde{S}_t)\sigma^2S_t^2(S_t^2f_s^2 + 2S_tf f_s) \right\} dt \end{aligned} \quad (4.3.4)$$

The martingale we use is

$$\begin{aligned} \lambda e^{-rt}EP(t, \tilde{S}_t) - \lambda \int_0^t e^{-ru} \{ \Delta(u, \tilde{S}_u)((f_s + \frac{1}{2}S_uf_{ss})S_u^2\sigma^2 + S_u^2(r-q)f_s + S_uf_u) \\ + \frac{1}{2}\Gamma(u, \tilde{S}_u)\sigma^2S_u^2(S_u^2f_s^2 + 2S_uf f_s) \} du - \lambda EP(0, \tilde{S}_0) \end{aligned} \quad (4.3.5)$$

Notice that we also have a weight, λ , for the martingale as we had for the discounted put in the previous section. We use a simple trapezoid sum for the integral in the simulation so that setting

$$\begin{aligned} I(t_i, S_{t_i}) &= e^{-rt_i} \left\{ \Delta(t_i, \tilde{S}_{t_i})((f_s(t_i, S_{t_i}) + \frac{1}{2}S_{t_i}f_{ss}(t_i, S_{t_i}))S_{t_i}^2\sigma^2 + S_{t_i}^2(r-q)f_s(t_i, S_{t_i}) \right. \\ &\quad \left. + S_{t_i}f_t(t_i, S_{t_i})) + \frac{1}{2}\Gamma(t_i, \tilde{S}_{t_i})\sigma^2S_{t_i}^2(S_{t_i}^2f_s^2(t_i, S_{t_i}) + 2S_{t_i}f(t_i, S_{t_i})f_s(t_i, S_{t_i})) \right\} \end{aligned} \quad (4.3.6)$$

where $\tilde{S}_{t_i} = g(t_i, S_{t_i})$.

for $i > 1$

$$M_{t_i} = \lambda e^{-rt_i} EP(t_i, \tilde{S}_{t_i}) - \lambda \left\{ \frac{1}{2} (I(0, S_0) + I(t_i, S_{t_i})) + \sum_{j=1}^{j=i-1} I(t_j, S_{t_j}) \right\} \Delta t - \lambda EP(0, \tilde{S}_0) \quad (4.3.7)$$

for $i = 1$

$$M_{t_i} = \lambda e^{-rt_i} EP(t_i, \tilde{S}_i) - \lambda \left\{ \frac{1}{2} (I(0, S_0) + I(t_i, S_{t_i})) \right\} \Delta t - \lambda EP(0, \tilde{S}_0) \quad (4.3.8)$$

and for $i = 0$

$$M_{t_i} = 0 \quad (4.3.9)$$

Now we have to determine a suitable choice for the function $g(t, S_t)$. We calculate American option prices using a Crank-Nicolson finite difference scheme with successive over relaxation from which we deduce S^* . The parameters we use are $T = 0.5$, $\sigma = 0.4$, $r = 0.06$, $K = 100$ and $q = 0$. We use 100 steps in the space direction and 50 steps in the time direction. We work with the variable $\log(S)$ and the grid is centered at $\log(K)$. The maximum value of $\log(S)$ for the grid is $\log(K) + 4\sigma\sqrt{T}$ and the minimum value of $\log(S)$ for the grid is $\log(K) - 4\sigma\sqrt{T}$. We calculate S^* values for each point on the grid. We do not really need much accuracy here since we are just looking for a functional form for $g(t, S)$ not exact parameter values. We will use the optimiser to find the parameter values. For fixed $t = 0$ and $t = 0.46$, we plot $\log(S^*)$ against $\log(S)$ (see figures (4-1) and (4-2)) and this seems like a reasonably linear relationship. At $t = 0.46$ we see the fit is worse than at $t = 0$. From figure (4-2) this seems to be mainly due to share values above the strike, but at high values of the share close to expiry, the option price is very small so the results from the finite difference scheme are unreliable. We will therefore not be too concerned about the fit close to expiry. We take $g(t, s) = a(t)s^{b(t)}$. We do a least squares fit within the range $\log(K) \pm 2\sigma\sqrt{T}$ to calculate values for $a(t)$ and $b(t)$ at each time step. We now need a functional form for $a(t)$ and $b(t)$. Figure (4-3) shows that up until about $t = 0.4$ we have a

reasonably linear relationship so we take

$$\begin{aligned} a(t) &= \alpha t + \beta, \\ b(t) &= \gamma t + \kappa \end{aligned} \tag{4.3.10}$$

We could take a more complicated functional form but we plan to optimise over the free parameters so we do not want too many of them!

4.3.1 Results

We use the optimiser to find values for λ , α , β , γ and κ that minimise the objective of price+standard deviation(sd). In the optimisation procedure we have 500 paths for the expectation (4.2.4). Once we have found values for the parameters of the martingale we price using 5,000 paths in the expectation. The optimisation procedure is done for each different starting S value. The results are shown below. Table (4.3) shows a reduction in MAD values and prices as compared to just using the discounted European put as the martingale (tables (4.1) and (4.2)), for the higher asset values the difference isn't as marked as for lower asset values. In table (4.4) we show the parameters used for the martingale. We see that $b(t)$ has a negative gradient and $a(t)$ has a positive gradient as in figure (4-3) although the fitted values at $S = 100$ are different. (4.5) shows results with the addition of the discounted put martingale to the hedging set with a weight λ_2 being added to the optimisation procedure for its weight, λ_1 referring to the weight of the martingale (4.3.5). This gives a slight improvement for the lower asset values. The optimal parameters tell us that we are long the martingale (4.3.5) and short the discounted put, the difference in weight is approximately 1. Table (4.7) shows the effect of varying the interest rate, r . We see that the hedge is better the lower the interest rate but even with $r = 0.1$ the MAD is still only 0.0803 which still is quite good. Table (4.9) shows the effect of varying σ . We see that the MAD gets worse as we increase σ but not as marked as in the case when we increased the interest rate. In all cases the MAD was below 0.052 which is quite good. The true American values in column 3 of table (4.7) and table (4.9) are also from AitSahlia and Carr (1997) where they use the average obtained from a 1000 and 1001 step binomial method as the exact values.

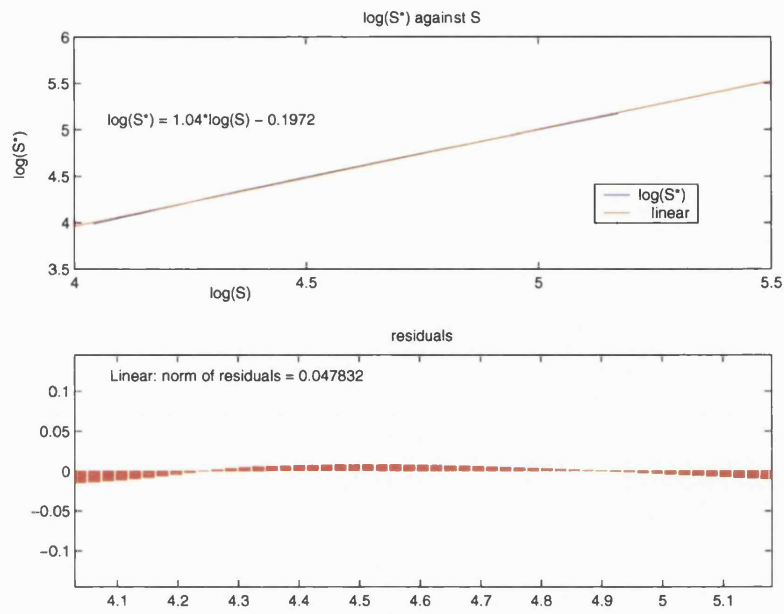


Figure 4-1: $\log(S^*)$ against $\log(S)$ at $t=0$

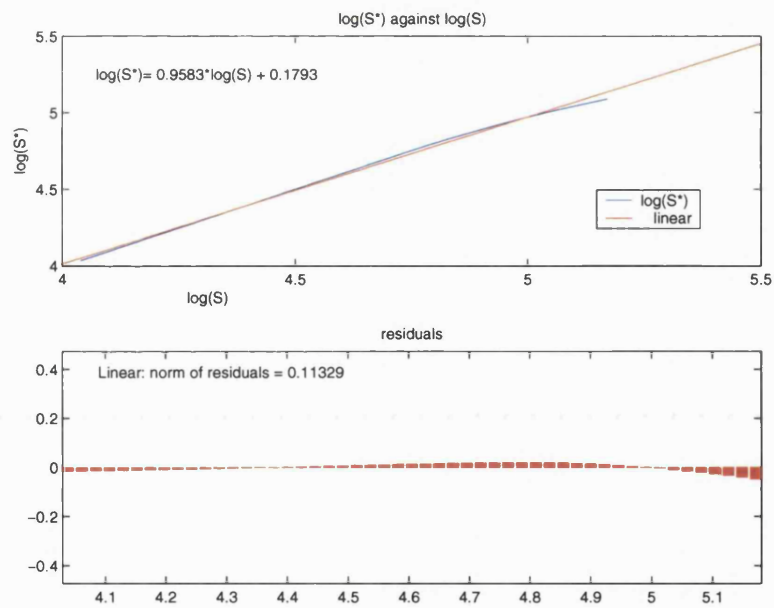


Figure 4-2: $\log(S^*)$ against $\log(S)$ at $t=0.46$

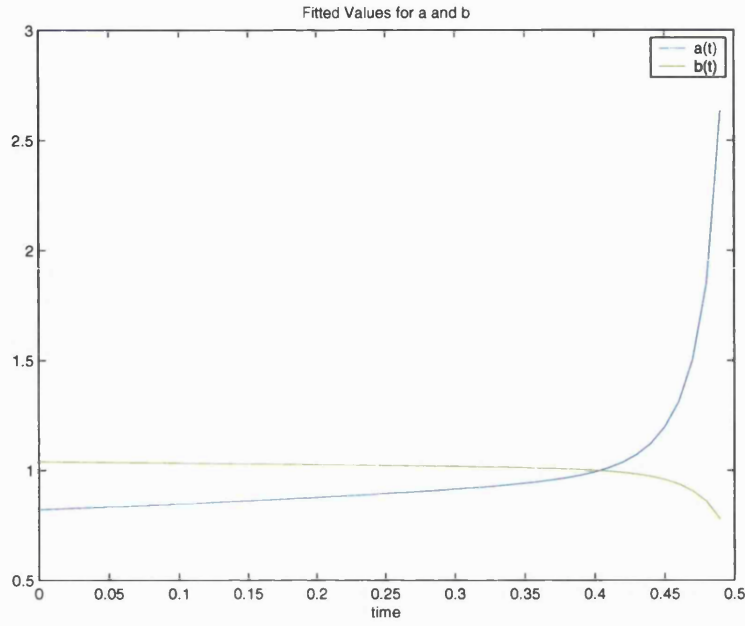


Figure 4-3: $a(t)$ and $b(t)$ against time

$S(0)$	American(True)	American (MC)	Standard error	MAD
80	21.6059	21.6324	0.0013	0.0692
85	18.0374	18.0626	0.0013	0.0714
90	14.9420	14.9415	0.0013	0.0704
95	12.2314	12.2516	0.0012	0.0638
100	9.9458	9.9649	0.0011	0.0544
105	8.0281	8.0420	0.0010	0.0468
110	6.4352	6.4456	0.0009	0.0401
115	5.1265	5.1338	0.0008	0.0342
120	4.0611	4.0670	0.0007	0.0290

Table 4.3: Simulation using discounted perturbed European put (4.3.5) as a martingale. Optimisation objective is to min (price +sd). 5000 sims

$S(0)$	α	β	$\gamma 10^{-1}$	κ	λ
80	0.50793	0.98030	-1.0514	1.0068	1.0824
85	0.45339	0.97415	-0.9554	1.0080	1.0753
90	0.41148	0.96992	-0.8752	1.0087	1.0697
95	0.39881	0.97809	-0.8448	1.0068	1.0697
100	0.36086	0.98139	-0.7683	1.0060	1.0666
105	0.33334	0.96661	-0.7178	1.0089	1.0584
110	0.30783	0.95742	-0.6680	1.0107	1.0520
115	0.29702	0.94812	-0.6483	1.0126	1.0475
120	0.25668	0.94505	-0.5617	1.0129	1.0411

Table 4.4: Parameters for simulation using discounted perturbed European put (4.3.5) as a martingale obtained using 500 paths

$S(0)$	American(True)	American (MC)	Standard error	MAD
80	21.6059	21.6240	0.0010	0.0551
85	18.0374	18.0550	0.0010	0.0573
90	14.9420	14.9341	0.0011	0.0583
95	12.2314	12.2453	0.0010	0.0547
100	9.9458	9.9599	0.0009	0.0484
105	8.0281	8.0395	0.0008	0.0440
110	6.4352	6.4441	0.0008	0.0398
115	5.1265	5.1334	0.0007	0.0342
120	4.0611	4.0682	0.0007	0.0307

Table 4.5: Simulation using discounted perturbed European put (4.3.5) and discounted European put as a martingale. Optimisation objective is to min (price +sd). 5000 paths

$S(0)$	$\alpha 10^{-1}$	β	$\gamma 10^{-1}$	κ	λ_1	λ_2
80	0.70605	0.99729	-0.1619	1.0011	8.8274	-7.7409
85	0.67099	0.99569	-0.1540	1.0014	8.6316	-7.5523
90	0.59284	0.99403	-0.1356	1.0017	9.4965	-8.4244
95	0.63790	0.99351	-0.1453	1.0018	8.5160	-7.4456
100	0.63159	0.99358	-0.1434	1.0018	7.8315	-6.7640
105	0.73438	0.99057	-0.1653	1.0025	6.1560	-5.0955
110	1.02247	0.98310	-0.2281	1.0042	4.0000	-2.9475
115	1.80138	0.96792	-0.3977	1.0077	2.0125	-0.9637
120	2.23490	0.95203	-0.4964	1.0113	1.23580	-0.1942

Table 4.6: Parameters for simulation using discounted perturbed European put (4.3.5) and discounted European put as a martingale, obtained using 500 paths

$S(0)$	r	American(True)	American (MC)	Standard error	MAD
100	0.02	10.7742	10.7782	0.0003	0.0151
100	0.04	10.3450	10.3537	0.0006	0.0323
100	0.06	9.9458	9.9599	0.0009	0.0484
100	0.08	9.95716	9.9591	0.0012	0.0640
100	0.1	9.92195	9.9246	0.0015	0.0803

Table 4.7: Simulation using discounted perturbed European put (4.3.5) and discounted European put as a martingale. Optimisation objective is to min (price +sd). 5000 paths

$S(0)$	r	$\alpha 10^{-1}$	β	$\gamma 10^{-1}$	κ	λ_1	λ_2
100	0.02	1.11688	0.99536	-0.25159	1.0018	1.0159	0.0050
100	0.04	0.73467	0.99303	-0.16652	1.0020	4.0437	-3.0037
100	0.06	0.63159	0.99358	-0.14339	1.0018	7.8315	-6.7640
100	0.08	0.56466	0.99272	-0.12791	1.0019	12.7448	-11.6511
100	0.1	0.11754	0.98492	-0.26339	1.0000	8.3848	-7.7256

Table 4.8: Parameters for simulation using discounted perturbed European put (4.3.5) and discounted European put as a martingale, obtained using 500 paths

$S(0)$	σ	American(True)	American (MC)	Standard error	MAD
100	0.30	7.2117	7.2252	0.0009	0.0458
100	0.35	8.5782	8.5921	0.0009	0.0468
100	0.40	9.9458	9.9599	0.0009	0.0484
100	0.45	11.3127	11.3270	0.0009	0.0499
100	0.50	12.6778	12.6922	0.0010	0.0514

Table 4.9: Simulation using discounted perturbed European put (4.3.5) and discounted European put as a martingale. Optimisation objective is to min (price +sd). 5000 paths

$S(0)$	σ	$\alpha 10^{-1}$	β	$\gamma 10^{-1}$	κ	λ_1	λ_2
100	0.30	0.56658	0.99178	-0.12667	1.0020	11.3633	-10.2761
100	0.35	0.53440	0.99391	-0.12047	1.0016	10.5471	-9.4699
100	0.40	0.63159	0.99358	-0.14339	1.0018	7.8315	-6.7640
100	0.45	0.61592	0.99403	-0.14086	1.0017	7.3310	-6.2714
100	0.50	0.10566	0.99074	-0.24135	1.0028	3.8429	-2.7896

Table 4.10: Parameters for simulation using discounted perturbed European put (4.3.5) and discounted European put as a martingale, obtained using 500 paths

4.4 Min-Put on 2 assets

In view of the success of the previous section we apply a similar approach to the problem of an American min-put on two assets. This option is a put option on the asset with the lowest value i.e. the process Z is

$$Z_t = \max_{i=1,2} e^{-rt} (K - S_i(t))^+ \quad (4.4.1)$$

where

$$S_i(t) = S_i(0) \exp(\sigma_i W_i(t) + (r - q_i - \sigma_i^2/2)t), \quad i = 1, 2. \quad (4.4.2)$$

where W_1 and W_2 are independent standard Brownian motions. (We set $\sigma_1 = \sigma_2 = \sigma$ for the purposes of numerical calculation).

Clearly an American min-put on two assets is worth at least as much as an American put on either asset. The greater the difference between the assets the closer the min-put price is to the American put price on the cheapest asset so we try $AP(t, S_1(t), S_2(t)) = \lambda EP(t, a(t) \phi_\epsilon(S_1(t), S_2(t))^{b(t)})$ where $\phi_\epsilon(S_1(t), S_2(t)) = \frac{1}{2}(S_1(t) + S_2(t) - \sqrt{\epsilon^2 + (S_1(t) - S_2(t))^2})$ and $AP(t, S_1(t), S_2(t))$ is an approximation to the American min-put price at time t . The function $\phi_\epsilon(S_1, S_2)$ is a smooth approximation to $S_1 \wedge S_2$. Let $\tilde{S}_t \equiv f(t, S_1(t), S_2(t)) \equiv a(t) \phi_\epsilon(S_1(t), S_2(t))^{b(t)}$ and by applying Ito's formula we get

$$\begin{aligned} d(e^{-rt} EP(t, \tilde{S}_t)) &= e^{-rt} (d(EP(t, \tilde{S}_t)) - r EP(t, \tilde{S}_t) dt) \\ &= e^{-rt} \left\{ \Delta(t, \tilde{S}_t) d\tilde{S}_t + \frac{1}{2} \Gamma(t, \tilde{S}_t) d\tilde{S}_t \cdot d\tilde{S}_t + \Theta(t, \tilde{S}_t) dt \right. \\ &\quad \left. - r EP(t, \tilde{S}_t) dt \right\} \end{aligned} \quad (4.4.3)$$

where

$$\begin{aligned} d\tilde{S}_t &= f_{S_1} dS_1(t) + \frac{1}{2} f_{S_1 S_1} dS_1(t) \cdot dS_1(t) + f_{S_2} dS_2(t) + \frac{1}{2} f_{S_2 S_2} dS_2(t) \cdot dS_2(t) + f_t dt \\ &= S_1(t) \sigma f_{S_1} dW_1(t) + S_2(t) \sigma f_{S_2} dW_2(t) + (S_1(t)(r - q_1) f_{S_1} + (r - q_2) S_2(t) f_{S_2} \\ &\quad + \frac{1}{2} S_1^2(t) \sigma^2 f_{S_1 S_1} + \frac{1}{2} S_2^2(t) \sigma^2 f_{S_2 S_2} + f_t) dt \end{aligned} \quad (4.4.4)$$

so

$$\begin{aligned}
d(e^{-rt}EP(t, \tilde{S}_t)) &\doteq e^{-rt} \left(\Delta(t, \tilde{S}_t)(S_1(t)(r - q_1)f_{S_1} + S_2(t)(r - q_2)f_{S_2} + \right. \\
&\frac{1}{2}S_1^2(t)\sigma^2 f_{S_1 S_1} + \frac{1}{2}S_2^2(t)\sigma^2 f_{S_2 S_2} + f_t) + \frac{1}{2}\Gamma(t, \tilde{S}_t)((S_1(t)\sigma f_{S_1})^2 + (S_2(t)\sigma f_{S_2})^2) \\
&\left. + \Theta(t, \tilde{S}_t) - rEP(t, \tilde{S}_t) \right) dt
\end{aligned} \tag{4.4.5}$$

The martingale we use is

$$\begin{aligned}
\lambda e^{-rt}EP(t, \tilde{S}_t) - \lambda \int_0^t e^{-ru} \left(\Delta(u, \tilde{S}_u)(S_1(u)(r - q_1)f_{S_1} + S_2(u)(r - q_2)f_{S_2} + \right. \\
\frac{1}{2}S_1^2(u)\sigma^2 f_{S_1 S_1} + \frac{1}{2}S_2^2(u)\sigma^2 f_{S_2 S_2} + f_u) + \frac{1}{2}\Gamma(u, \tilde{S}_u)((S_1(u)\sigma f_{S_1})^2 + (S_2(u)\sigma f_{S_2})^2) + \\
\left. \Theta(u, \tilde{S}_u) - rEP(u, \tilde{S}_u) \right) du - \lambda EP(0, \tilde{S}_0)
\end{aligned} \tag{4.4.6}$$

In the Monte Carlo simulation both asset price processes in (4.4.2) are simulated in the same way as the single asset in (4.2.2) using 50 timesteps. At time t_i (defined as in section 4.2) the process Z_{t_i} is given by

$$Z_{t_i} = \max_{j=1,2} e^{-rt_i} (K - S_j(t_i))^+ \tag{4.4.7}$$

In calculating the integral in (4.4.6) we use a trapezoid rule as we did for the refined martingale of section (4.3). We use 50 timesteps with a Richardson extrapolation as we did for the American put on a single asset and 10,000 simulations for the expectation. When using the optimiser to search for optimal martingale parameters we use only 500 paths in the expectation. Parameter values are $T = 0.5$, $r = 0.06$, $\sigma_1 = \sigma_2 = 0.6$, $K = 100$ and $q_1 = q_2 = 0$.

4.4.1 Results

Table (4.11) uses the martingale proposed by Rogers (2002), the martingale increments are given by the discounted European put of the asset which is the cheapest (optimising over λ for its weight). The American (FD) values in column 3 are taken from Rogers (2002). In table (4.11) the objective of the optimization procedure is to minimize the price. In table (4.12) the objective of the optimi-

sation procedure is to minimize the price + standard deviation. This results in a slight improvement in the MAD values but worse prices. Table (4.13) shows the results of using the martingale given by (4.4.6). We have 6 parameters to optimise over λ , α , β , γ , κ and ε . As compared to table (4.12) this shows a substantial reduction in the MAD values. Rogers (2002) introduces an exchange type martingale in addition to the discounted European put on the cheapest asset i.e. assuming the option is in the money and assuming $S_2 \geq S_1$ the martingale increments are given by the option which pays $(S_1(T) - S_2(T))^+$ at expiry. When adding this exchange type martingale we include an additional parameter in the optimisation procedure for its weight λ_2 , λ_1 refers to the weight of the martingale from (4.4.6). Table (4.15) shows the most improvement in the prices for the (80, 100) and (80, 120) cases where the weights for the exchange martingale are much larger than in the other cases in table (4.16).

4.5 Conclusions

In this chapter we have looked at the dual method of pricing American options. In particular we have looked at the problem of pricing an American put on a single asset and an American min-put on two assets. We refined the discounted European put option used by Rogers (2002) as a hedging martingale for the American put to get better prices and hedges. We used a similar technique to get a hedging martingale for the American min-put on two assets. This resulted in a substantial reduction in MAD values but gave worse prices than simply using the discounted European put on the cheapest asset.

For pricing American options, this technique of replacing the share price(s) in a formula for the corresponding European option for a function of the share price(s) can only be done when a closed form European option formula is available. The utility of the dual method over primal methods (such as dynamic programming) is when dealing with high dimensional options. Here it would be difficult to apply the technique used in this chapter since a closed form solution may not be available and you would need a good idea of the behaviour of the American option as compared to the European option. We get better results with the put than the min-put.

$S_1(0)$	$S_2(0)$	American(FD)	American (MC)	SE	MAD	λ
80	80	37.30	37.68	0.093	7.3003	0.9971
80	100	32.08	32.52	0.083	6.5615	1.014
80	120	29.14	29.42	0.065	4.7943	1.0241
100	100	25.06	25.20	0.083	6.3468	1.0153
100	120	20.91	21.10	0.071	5.3388	1.0188
120	120	15.92	15.97	0.064	4.7136	1.0136

Table 4.11: Min-Put on 2 Assets. Martingale is the discounted European put on the cheapest asset. Optimisation objective is to min (price).

$S_1(0)$	$S_2(0)$	American(FD)	American (MC)	SE	MAD	λ
80	80	37.30	38.57	0.078	6.0272	0.8416
80	100	32.08	33.07	0.073	5.5902	0.8970
80	120	29.14	29.68	0.060	4.2241	0.9513
100	100	25.06	25.48	0.077	5.8552	0.9365
100	120	20.91	21.19	0.067	5.0653	0.9709
120	120	15.92	16.05	0.062	4.5639	0.9741

Table 4.12: Min-Put on 2 Assets. Martingale is the discounted European put on the cheapest asset. Optimisation objective is to min (price+sd).

$S_1(0)$	$S_2(0)$	American(FD)	American (MC)	SE	MAD
80	80	37.30	38.25	0.018	1.4169
80	100	32.08	33.04	0.019	1.4677
80	120	29.14	29.97	0.018	1.4012
100	100	25.06	25.89	0.019	1.5264
100	120	20.91	21.69	0.019	1.4835
120	120	15.92	16.53	0.017	1.3801

Table 4.13: Min-Put on 2 Assets. Martingale from (4.4.6). Optimisation objective is to min (price+sd).

$S_1(0)$	$S_2(0)$	α	β	γ	κ	λ	ε
80	80	-0.40890	1.31269	0.16396	0.9062	0.9934	25.5569
80	100	-0.62560	1.37950	0.20019	0.8981	0.9882	28.2605
80	120	-0.51618	1.36306	0.15691	0.9111	0.9940	28.7373
100	100	-0.38162	1.24894	0.15364	0.9249	0.9827	28.4731
100	120	-0.83035	1.37377	0.22633	0.9068	0.9794	30.7711
120	120	-0.80638	1.39755	0.21422	0.9049	0.9922	30.0481

Table 4.14: Min-Put on 2 Assets, parameters for martingale from (4.4.6). Optimisation objective is to min (price+sd).

$S_1(0)$	$S_2(0)$	American(FD)	American (MC)	SE	MAD
80	80	37.30	38.16	0.017	1.3500
80	100	32.08	32.84	0.018	1.4634
80	120	29.14	29.78	0.017	1.2850
100	100	25.06	25.83	0.018	1.4396
100	120	20.91	21.57	0.018	1.4042
120	120	15.92	16.53	0.016	1.2938

Table 4.15: Min-Put on 2 Assets. Martingale (4.4.6) and exchange martingale. Optimisation objective is to min (price+sd)

$S_1(0)$	$S_2(0)$	α	β	γ	κ	λ_1	λ_2	ε
80	80	0.13208	1.08768	0.06317	0.94550	1.0040	0.0666	23.478
80	100	0.42557	0.99092	0.00391	0.96469	1.0153	0.1297	23.1507
80	120	-0.24951	1.23349	0.09352	0.93515	1.0169	0.1274	23.5627
100	100	0.02384	1.10817	0.08204	0.94593	0.9985	0.0550	26.4073
100	120	-0.11620	1.10997	0.09809	0.94980	0.9947	0.0842	26.1742
120	120	-0.08364	1.26205	0.07564	0.92855	1.0154	0.0360	28.2731

Table 4.16: Min-Put on 2 Assets, parameters for martingale from (4.4.6) and exchange type martingale.

Appendix A

Appendix to chapter 2

Proposition 1. If $w_t = H_t S_t + K_t \geq 0$ and $K_t < 0$ then $\mathbb{P}(w_s \geq 0; \forall s \geq t) < 1$

Proof. Suppose

$$\sup_{t \leq u \leq t+\delta} S_u \leq 2S_t, \text{ and}$$

$$\frac{1}{2}c \leq S_u \leq 2c \text{ for } \tau \leq u \leq t + \delta$$

where $\tau = \inf\{u > t : S_u = c\} \wedge (t + \delta)$, $\delta > 0$, $c > 0$ and c is chosen such that $H_t(2c) + K_t + \delta \frac{2c}{\varepsilon} < 0$.

$K_{t+\delta} - K_t = -\int_t^{t+\delta} S_u h_u du - \varepsilon \int_t^{t+\delta} S_u h_u^2 du$ and $w_{t+\delta} = H_{t+\delta} S_{t+\delta} + K_{t+\delta}$ so that

$$\begin{aligned} w_{t+\delta} &= H_{t+\delta} S_{t+\delta} + K_t + \int_t^{t+\delta} (-S_u h_u - \varepsilon S_u h_u^2) du \\ &= H_t S_{t+\delta} + K_t + \int_t^{t+\delta} (S_u (-h_u - \varepsilon h_u^2) + S_{t+\delta} h_u) du \\ &\leq H_t S_{t+\delta} + K_t + \int_t^{t+\delta} \frac{1}{4} \frac{(S_{t+\delta} - S_u)^2}{\varepsilon S_u} du \\ &\leq H_t S_{t+\delta} + K_t + \int_t^\tau \frac{S_t^2}{\varepsilon c} du + \int_\tau^{t+\delta} \frac{2c}{\varepsilon} du \\ &\leq H_t S_{t+\delta} + K_t + (\tau - t) \frac{S_t^2}{\varepsilon c} + \delta \frac{2c}{\varepsilon} \\ &< 0 \end{aligned} \tag{A.0.1}$$

on ε set of positive probability since $\mathbb{P}(t < \tau < a) > 0$ for any $a > t$

Proposition 2. If $w_t = H_t S_t + K_t \geq 0$ and $H_t < 0$ then $\mathbb{P}(w_s \geq 0; \forall s \geq t) < 1$

Proof. Suppose

$$\inf_{t \leq u \leq t+\delta} S_u \geq \frac{1}{2} S_t, \text{ and}$$

$$c - d \leq S_u \leq c + d \text{ for } \tau \leq u \leq t + \delta$$

where $\tau = \inf\{u > t : S_u = c\} \wedge (t + \delta)$, $c > d > 0$, $\delta > 0$ and c is chosen such that $H_t(c - d) + K_t + \delta \frac{d^2}{\varepsilon(c-d)} < 0$.

$K_{t+\delta} - K_t = -\int_t^{t+\delta} S_u h_u du - \varepsilon \int_t^{t+\delta} S_u h_u^2 du$ and $w_{t+\delta} = H_{t+\delta} S_{t+\delta} + K_{t+\delta}$ so that

$$\begin{aligned} w_{t+\delta} &= H_{t+\delta} S_{t+\delta} + K_t + \int_t^{t+\delta} S_u (-h_u - \varepsilon h_u^2) du \\ &= H_t S_{t+\delta} + K_t + \int_t^{t+\delta} (S_u (-h_u - \varepsilon h_u^2) + S_{t+\delta} h_u) du \\ &\leq H_t S_{t+\delta} + K_t + \int_t^{t+\delta} \frac{1}{4} \frac{(S_{t+\delta} - S_u)^2}{\varepsilon S_u} du \\ &\leq H_t S_{t+\delta} + K_t + \int_t^\tau \frac{1}{2} \frac{(c+d)^2}{\varepsilon S_t} du + \int_\tau^{t+\delta} \frac{d^2}{\varepsilon(c-d)} du \\ &\leq H_t S_{t+\delta} + K_t + (\tau - t) \frac{1}{2} \frac{(c+d)^2}{\varepsilon S_t} + \delta \frac{d^2}{\varepsilon(c-d)} \\ &< 0 \end{aligned} \tag{A.0.2}$$

on ε set of positive probability since $\mathbb{P}(t < \tau < a) > 0$ for any $a > t$

Solution to the corresponding finite horizon problem of section (2.6).

$$dU_t = U_t(\sigma(1 - \pi_*) dW_t' + \rho' dt) + \alpha_t dt \quad (\text{A.0.3})$$

with the objective

$$G(U, t; T) = \inf_{\alpha} \mathbb{E}_{\mathbb{Q}} \left[\int_t^T \gamma^{-R} e^{-\rho' t} (\epsilon \alpha_t^2 + \frac{1}{2} \sigma^2 R (\pi_* - U_t)^2) dt \mid U_t = U \right] \quad (\text{A.0.4})$$

where $\rho' > 0$.

We conjecture a quadratic solution in U to solve the corresponding HJB equation. This procedure is proved rigorously in Yong and Zhou (1999).

$$G(U, t) = A(t)U^2 + B(t)U + C(t) \quad (\text{A.0.5})$$

The HJB equation for this problem is

$$\begin{aligned} & \inf_{\alpha} \left\{ \exp(-\rho' t) \gamma^{-R} (\epsilon \alpha^2 + \frac{1}{2} \sigma^2 R (\pi_* - U)^2) + (\rho' U + \alpha) G_U + \right. \\ & \left. \frac{1}{2} \sigma^2 (1 - \pi_*)^2 U^2 G_{UU} + G_t \right\} = 0 \end{aligned} \quad (\text{A.0.6})$$

which gives the non-linear pde

$$\begin{aligned} & -G_U(U, t)^2 + e^{-\rho' t} \left\{ \frac{a(\pi_* - U)^2}{e^{\rho' t}} + \frac{b G_t(U, t)}{\rho'} + b U G_U(U, t) + c U^2 G_{UU}(U, t) \right\} \\ & = 0 \end{aligned} \quad (\text{A.0.7})$$

where we have used the definition of a, b and c from section (2.6). Substituting (A.0.5) into (A.0.7) gives the system of equations

$$\begin{aligned} & \frac{a}{e^{2t\rho'}} + \frac{2bA(t)}{e^{t\rho'}} + \frac{2cA(t)}{e^{t\rho'}} - 4A(t)^2 + \frac{bA'(t)}{e^{t\rho'}\rho'} = 0 \\ & \frac{-2a\pi_*}{e^{2t\rho'}} + \frac{bB(t)}{e^{t\rho'}} - 4A(t)B(t) + \frac{bB'(t)}{e^{t\rho'}\rho'} = 0 \\ & \frac{a\pi_*^2}{e^{2t\rho'}} - B(t)^2 + \frac{bC'(t)}{e^{t\rho'}\rho'} = 0 \end{aligned} \quad (\text{A.0.8})$$

With the boundary conditions $A(T) = B(T) = C(T) = 0$ the solution is

$$A(t) = \frac{1}{8} e^{-\rho' t} \left(b + 2c + \eta + \frac{8\eta}{-4 + \frac{4(b+2c-\eta)e^{\frac{(T-t)\eta\rho'}{b}}}{(b+2c+\eta)}} \right) \quad (\text{A.0.9})$$

$$B(t) = \frac{4a \left(- \left(e^{\frac{T\eta\rho'}{b}} (b+2c-\eta)^2 \right) - 4(b+2c) e^{\frac{((b+2c)(t-T)+(t+T)\eta)\rho'}{2b}} \eta + e^{\frac{t\eta\rho'}{b}} (b+2c+\eta)^2 \right) \pi_*}{e^{t\rho'} (b+2c-\eta) (b+2c+\eta) \left(e^{\frac{T\eta\rho'}{b}} (-b-2c+\eta) + e^{\frac{t\eta\rho'}{b}} (b+2c+\eta) \right)} \quad (\text{A.0.10})$$

$$C(t) = \frac{a\pi_*^2}{b} (e^{-\rho't} - e^{-\rho'T}) - \int_t^T \frac{\rho'}{b} e^{\rho's} B^2(s) ds \quad (\text{A.0.11})$$

where $\eta = \sqrt{16a + (b+2c)^2}$

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